INSA de Rouen

STPI - SIB - M2

Tutorial n$^{\text{o}}$ 4
Combinatorics. Abstract algebraic notions - basics.

Exercise 1 *Induction and recurrence relations.*

1. *Prove the Pascal's "triangle" identity:* $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$*, for values of* k *and* n *when all the terms are defined. Explain its combinatorial meaning.*

Deduce the the Newton's binomial formula: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

Compute the following sums: $\sum_{k=0}^{n} \binom{n}{k}$, $\sum_{k=0}^{n} (-1)^k \binom{n}{k}$, $\sum_{k=0}^{[n/2]} \binom{n}{2k}$.

2. *Prove (or recall the proof of) the Moivre's formula:* $(\cos\alpha + i\sin\alpha)^n = \cos(n\alpha) + i\sin(n\alpha)$. *Deduce the formulas for* $\cos(n\alpha)$ *and* $\sin(n\alpha)$.

3. *Simplify the expression* $(k+1)^2 - k^2$. *Construct and prove the formula for* $\sum_{k=0}^{n} k$.

Perform a similar construction for $\sum_{k=0}^{n} k^2$ *and* $\sum_{k=0}^{n} k^3$

4. *Compute the sum of the angles of a convex / non-convex planar polygone.*

Exercise 2 *Combinatorial geometry*

1. *Consider points* (x,y) *on the plane, with coordinates taking values in the set* $\{-1,0,1\}$. *How many different couples of them can we form? How many non-aligned triples? How many distinct lines pass through at least two of these points?*

2. *Consider points* (x,y,z) *in the* 3*-dimensional space, with coordinates taking values in the set* $\{-1,0,1\}$. *How many different couples/triples of them can we form? How many non-aligned triples? How many distinct lines pass through at least two of these points?* How many distinct planes pass through at least three of these points?**

Exercise 3 *Algebraic structures*

1. *Consider* $E = \{\heartsuit, \diamondsuit, \spadesuit\}$ *– a set of three elements, equipped with a product* $*\colon E\times E \to E$, *given by the following table:*

$*$	$\heartsuit$	$\diamondsuit$	$\spadesuit$
$\heartsuit$	$\diamondsuit$	$\heartsuit$	$\heartsuit$
$\diamondsuit$	$\diamondsuit$	$\spadesuit$	$\diamondsuit$
$\spadesuit$	$\spadesuit$	$\spadesuit$	$\heartsuit$

.

Compute the following expressions:

$(\heartsuit * \diamondsuit) * \spadesuit$ $\quad$ $(\heartsuit * \diamondsuit) * \heartsuit$ $\quad$ $\heartsuit * (\heartsuit * \diamondsuit)$ $\quad$ $\spadesuit * (\spadesuit * (\spadesuit * \spadesuit))$

$\heartsuit * (\diamondsuit * \spadesuit)$ $\quad$ $\heartsuit * (\diamondsuit * \heartsuit)$ $\quad$ $(\spadesuit * \spadesuit) * (\spadesuit * \spadesuit)$ $\quad$ $\spadesuit * ((\spadesuit * \spadesuit) * \spadesuit)$

From the results of the computation decide if E *with the the defined operation is associative/commutative.*

2. *How many binary operations can we define on a set of* n *elements?*

3. *In the table of a binary operation, what is the characteristic property of a line/column corresponding to the neutral element? To an invertible element? Describe up to isomorphism all the groups of two / three elements. Which of them are abelian?*