INSA de Rouen

STPI - SIB - M2

Tutorial n$^{\mathrm{o}}$ 3
Geometry of $\mathbb{R}^2$ and $\mathbb{R}^3$

Exercise 1 *Scalar product and center of mass in $\mathbb{R}^2$.*

1. *Let ABC be a triangle. Prove that for any point M the following relation holds true:*

$$\overrightarrow{MA} \cdot \overrightarrow{BC} + \overrightarrow{MB} \cdot \overrightarrow{CA} + \overrightarrow{MC} \cdot \overrightarrow{AB} = 0.$$

In the French literature it is called Stewart equality. Use this result to prove that three altitudes of any triangle intersect in the same point (called the orthocenter of the triangle).

2. *Prove that three medians of any triangle intersect in the same point (called the centroid of the triangle). Recall the physical and geometric interpretation of this fact.*

3. *Recall the solutions of the equation $z^6 = 1$ in $\mathbb{C}$. Compute their sum.*
Compute the sum of the solutions of the equation $z^7 = 1$ in $\mathbb{C}$.
Does a similar result hold for the solutions of the equation $z^n = 1$ in $\mathbb{C}$ for arbitrary n?

Exercise 2 *Metric geometry in $\mathbb{R}^2$.*
Consider in $\mathbb{R}^2$ the points $O(0,0)$, $A(1,3)$, $B(3,2)$, $C(3,4)$.

1. *Give a parametric description of a line L_1 passing through A and B; give an algebraic (cartesian) equation of L_1 in the form $ax + by + c = 0 \Leftrightarrow \vec{r} \cdot \vec{n} = -c$, where $\vec{n} \in \mathbb{R}^2$ and $c \in \mathbb{R}$.*

2. *Compute the coordinates of the projection H of A to the line containing BC, and of the point D which is the image of A by a reflection with respect to BC. Compute the coordinates of the point E which is the image of A by a central symmetry with respect to C. Compute the area of the quadrangle $ABDE$.*

3. *Give a parametric and an algebraic description of the orthogonal bisector of the segment AB: $L_2 \perp L_1$, passing through the center O' of AB.*

4. *Find the intersection of L_2 and the line L_3 passing through B and C.*

5. *Consider L_1 and L_2 as new coordinate axes. Find the coordinates of C, D, E and H in this coordinate system, i.e. find the lengths of the projections of $O'C$, $O'D$, $O'E$, $O'H$ to L_1 and L_2.*

Exercise 3 *Metric geometry in $\mathbb{R}^3$.*
Consider in $\mathbb{R}^3$ the points $A(-1,6,3)$, $B(3,0,-1)$, $C(3,-1,0)$, $D(6,7,3)$.

1. *Give a parametric description of the plane P containing the points A, B, C.*
Give an algebraic (cartesian) equation of this plane P in the form
$ax + by + cz + d = 0 \Leftrightarrow \vec{r} \cdot \vec{n} = -d$, *where $\vec{n} \in \mathbb{R}^3$ and $d \in \mathbb{R}$*

2. *Show that the point $E(8,5,5)$ is at the same distance from the points A, B, C.*

3. *Let L be a straight line orthogonal to P, passing through the point E. Give a parametric description of this line in the form $\vec{r} = \vec{v} + \lambda \vec{w}$.*

4. *Determine the point M on the line L such that the distance DM is equal to the distance AM.*

5. *Check that the distances for the point M to the points A, B, C, D are the same and equal to 5.*

6. *Compute the volumes of the tetrahedra $ABCD$, $ABCM$, $ABCE$.*
Compare the ratios of the volumes with the ratios of the distances from D, M, and E to P.
Deduce the area of the triangle ABC.

7. *Compute the distance between the lines passing through A and B, and C and E respectively.*

Exercise 4 *Systems of linear equations.*

1. *Give the algebraic equations of the following planes.*

$$P_1 = \left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix} \right\} \qquad P_2 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} \right\}$$

$$P_3 = \left\{ \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix} \right\} \qquad P_4 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

2. *Describe the intersection between* *a). P_1, P_2 and P_3,* *b). P_1, P_2 and P_4*

Exercise 5 *Optional – if time permits.*
Consider the linear mapping $\mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix.

$$M_D = \frac{1}{9} \begin{pmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \end{pmatrix}.$$

1. *Check that $M_D^2 = M_D = M_D^T$.*

2. *Describe the kernel of this mapping*

$$Ker(M_D) = \{\vec{v} \in \mathbb{R}^3 : M_D \vec{v} = \vec{0}\}.$$

3. *Describe the orthogonal complement to this kernel*

$$Ker(M_D)^{\perp} := \{\vec{w} \in \mathbb{R}^3 : \vec{w} \cdot \vec{v} = 0 \, \forall \, \vec{v} \in Ker(M_D)\}$$

Show that for any vector w in this orthogonal complement there exists a $\lambda \neq 0$ such that $M_D \vec{w} = \lambda \vec{w}$.

4. *Deduce that the mapping $\vec{v} \mapsto M_D \vec{v}$ is an orthogonal projection to the set $D \subset \mathbb{R}^3$ to be specified.*
Consider now the linear mapping $\mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix $M_P = I - M_D$.

5. *Check that $M_P^2 = M_P = M_P^T$.*

6. *Describe the kernel of this mapping and its orthogonal complement.*

7. *Explain the geometric meaning of this mapping and compare it with the one of M_D.*