INSA de Rouen

STPI - SIB - M2

Tutorial n$^{\mathrm{o}}$ 2
Complex numbers and geometry of $\mathbb{R}^2$

In this tutorial we identify the set of complex numbers $\mathbb{C}$ with a set of points in $\mathbb{R}^2$:
$\mathbb{C} \ni z = x + iy \leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$

Exercise 1

Describe and draw the sets of points in $\mathbb{R}^2$ defined by the following relations:

a). $|(1-i)z - 3i| = 3$; *b).* $|1 - z| \le \frac{1}{2}$; *c).* $\Re\mathrm{e}(1 - z) \le \frac{1}{2}$;

d). $\Re\mathrm{e}(iz) \le \frac{1}{2}$; *e).* $\Im\mathrm{m}(iz) \le \frac{1}{2}$; *f).* $\left|\frac{z-5}{z-3}\right| = 1$.

Exercise 2

1. *Describe and draw the image of the unit circle, the coordinate axes, and arbitrary lines passing through the origin for the mappings $\mathbb{R}^2 \to \mathbb{R}^2$ associated to the following operations.*
Explain the geometric meaning of these mappings.

a). $z \mapsto iz$; *b).* $z \mapsto e^{i\varphi} z$, $\varphi \in \mathbb{R}$; *c).* $z \mapsto rz$, $r \in \mathbb{R}$; *d).* $z \mapsto re^{i\varphi} z$, $r, \varphi \in \mathbb{R}$;

e). $z \mapsto \bar{z}$; *f).** $z \mapsto az + b$, $a, b \in \mathbb{C}$; *g).** $z \mapsto a\bar{z} + b$, $a, b \in \mathbb{C}$.

2. *Give the analytic expression of the following mappings $\mathbb{R}^2 \to \mathbb{R}^2$, i.e. describe them as mappings $\mathbb{C} \to \mathbb{C}$: $z \mapsto f(z)$. Describe the inverse to these mappings as well.*

a). Parallel transport (translation) along a vector $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$;

b). Point reflection with respect to a point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$;

c). Rotation around a point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ *by an angle* φ;

d). Rotational homothecy with a center at a point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ *with an angle* φ *and a coefficient* r;

e). Reflection symmetries with respect to coordinate axes;

f). Reflection symmetry with respect to an arbitrary line passing through the origin;

g). Reflection symmetry with respect to an arbitrary line not passing through the origin.

Exercise 3

1. *From the mappings in the previous exercise choose the isometries, i.e. those that preserve the distance between points.*

2. *Consider now $\mathbb{R}^2$ as a vector space.*

a). From the mappings in the previous exercise choose those that are linear.

b). Fixing the canonical basis $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ *of $\mathbb{R}^2$ construct the matrices of these mappings.*

c). Prove that a composition of two rotations around the origin is again a rotation
(There are two ways to do it: using the analytic expression or matrix multiplication).