Integrable Hamiltonian systems arising as averaged equations for geodesic flows on weakly deformed spheres

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Basic examples of shortest lines:

on the plane - a straight line,
on the sphere - a great circle
– known from Antiquity.

The general problem of geodesic lines:

J. Bernoulli - posing the question about the shortest lines on arbitrary surfaces;

L. Euler, "De linea brevissima..."(On the shortest line...), 1728, - the equations for geodesics on 2D surfaces.

Some integrable cases:

surfaces of revolution - A. Clairaut, 1743;
ellipsoid - J. Jacobi, 1839;
metrics corresponding to rigid body motion
- A. Bolsinov, V. Kozlov, A. Fomenko, 1995.

General case:

For the majority of surfaces there is no known exact solution, and very little is known about geodesics.

Methods of the perturbation theory:

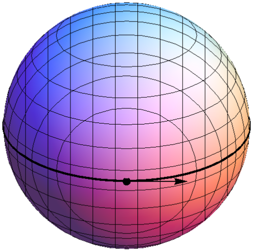
Study of systems close to the known integrable ones.

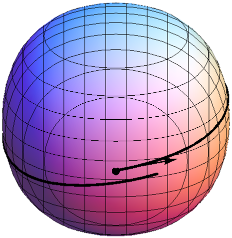
The idea of applying asymptotic methods to the study of geodesics:

H. Poincaré, Sur les lignes geodesique des surfaces convexes, Trans. Amer. Math. Soc. 6 (1905), 237-274.
Poincaré studied closed geodesics.
We aim at applying the same method to all geodesics, not only the closed ones.

The main idea:

Consider a surface obtained by a small perturbation of the standard sphere.
While for the standard sphere geodesics are great circles, for the deformed sphere a geodesic will be close to a planar section, but with a small shift:





Asymptotic description:

The motion consists of a fast component - rotation around the surface, and a slow component - the oscillation of the plane approximating the current loop of the curve.

The method of averaging:

Obtaining an averaged system for the slow variables, asymptotically describing their motion in the initial system.

The concrete formulation of the problem :

We consider geodesic lines on the surfaces close the standard $(n-1)$-dimensional sphere, and defined by the equation:

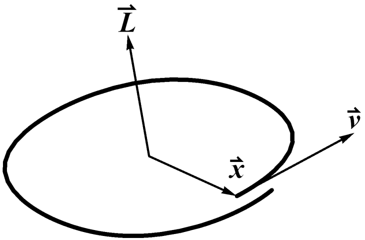
$$\varphi(\vec{x}) \equiv \sum_{i=1}^{n} x_i^2 - 1 + \varepsilon\,\psi(\vec{x}) = 0, \qquad \varepsilon \ll 1$$

We describe the geodesics in Cartesian coordinates, using the Lagrange equations of the first kind:

$$\ddot{\vec{x}} = \lambda\,\frac{\partial \varphi}{\partial \vec{x}}.$$

How to describe the slow motion?

In the 3D case we need to describe the position of the plane approximating the current loop of the geodesic. The plane can be identified by its normal vector, which is given by the angular momentum:



The multidimensional case:

For a motion in $\mathbb{R}^n$ the angular momentum is a skew-symmetric matrix:

$$l_{ij} = x_i \dot{x}_j - x_j \dot{x}_i, \quad i, j = 1 \dots n.$$

Note that the Poisson brackets of these quantities form the Lie-Poisson algebra $so(n)$.
For the standard sphere any geodesic is a great circle and lies in a fixed plane, so $l_{ij} = const$.
For a deformed sphere these variables slowly change with time.

The dynamics on the Grassmanian:

It is important to note that l_{ij} are in fact the Plücker coordinates of the 2-plane containing the vectors $\vec{x}, \dot{\vec{x}}$.
Indeed, the Plücker coordinates of the 2-plane with a basis of two vectors $\vec{a}, \vec{b}$ are defined as:

$$a_i\, b_j - a_j\, b_i, \quad i, j = 1 \ldots n,$$

which coincides with the definition of angular momentum.

Thus, the dynamics of l_{ij} defines a slowly moving point on the Grassmanian $G(2, n)$ of all 2-planes containing the origin.

The method of averaging:

The construction of a system for the slow variables l_{ij} whose solutions are close to the trajectories of these variables in the system for the geodesics.

Asymptotic reduction:

The system of equations for geodesics, with phase dimension $2n-2$

$$\Downarrow$$

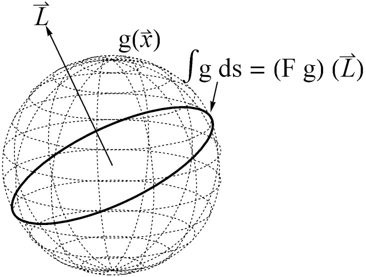
The averaged system on the Grassmanian G(2, n), with phase dimension $2n-4$

Integral geometry:

The study of integral transforms obtained by integrating a function on a manifold over all its submanifolds of a given type.

Examples of transforms:

Minkowski-Funk transform - take a function on the standard 2D sphere and integrate it over all great circles:



Other classical transforms:

Radon transform - take a function on the 2D plane and integrate it over all straight lines.

Generalized Radon transform - take a function in $\mathbb{R}^n$ and integrate it over all hyperplanes.

John (X-ray) transform - take a function in $\mathbb{R}^n$ and integrate it over all 1D-lines.

John (X-ray) transform on the sphere - take a function on the standard multidimensional sphere and integrate it over all great circles (cross-sections by 2-planes through the origin):

$$(Jf)(\hat{l}) = \int_0^{2\pi} f\left(\cos t \ \vec{e}_1(\hat{l}) + \sin t \ \vec{e}_2(\hat{l})\right) \, dt.$$

Further generalized by Gelfand and Granin.

The connection of averaging and integral geometry:

Theorem. The averaged system for the slow variables l_{ij} in the system describing the geodesics on a deformed sphere is a Hamiltonian system on the Grassmanian $G(2, n)$ with the Hamiltonian:

$$H(\hat{l}) = \frac{\varepsilon}{2\pi} J\,\psi,$$

and the Poisson algebra $so(n)$. Here J is the X-ray transform on the sphere.

Why is this helpful?

The connection of the reduction procedure with integral geometry allows one to:

- effectively compute the Hamiltonians of the averaged systems (which would be very hard using the equations for geodesics in local coordinates on the surface, even for relatively simple surfaces);

- transfer the results of integral geometry, obtaining properties of the averaged systems.

Transferring the results of integral geometry:

Property. The X-ray transform on the 2D sphere sends any even polynomial into an even polynomial of the same degree.

Corollary. For the case of S^2, if the deformation function $\psi(\vec{x})$ is an even polynomial then the Hamiltonian of the averaged system is also an even polynomial of the same degree in the components of the momentum l_{ij}.

Transferring the results of integral geometry:

F. John showed that not any function on G(2,4) can be obtained as the X-ray transform of some function on the 3D sphere. For this he found the following criterion.

Property. A function f on G(2,4) is in the image of the X-ray transform iff it satisfies the John ultrahyperbolic equation:

$$\frac{\partial^2 f}{\partial l_{12}\partial l_{34}} - \frac{\partial^2 f}{\partial l_{13}\partial l_{24}} + \frac{\partial^2 f}{\partial l_{14}\partial l_{23}} = 0.$$

Corollary. For the case of S^3 the Hamiltonian of the averaged system satisfies the above John ultrahyperbolic equation.

Integrable systems on Lie algebras:

There has been extensive study of integrable Hamiltonian systems on so(n) from the XIX century up to the recent years. Most of the research is concerned with Hamiltonians of degree 2, which have a mechanical interpretation in terms of rigid bodies in $\mathbb{R}^n$ or rigid bodies with a cavity filled with an ideal fluid (the integrable cases of Poincaré, Manakov, Steklov, Adler – Van Moerbeke, Sokolov – Borisov – Mamaev).

However, the above connection with the problem of geodesics stresses the importance of studying integrable systems on so(n) with Hamiltonians of arbitrary degree. Indeed, for any integrable Hamiltonian on so(n) satisfying the conditions defining the image of the X-ray transform, the inverse X-ray transform can be applied to obtain a deformed sphere with an integrable averaged system for the geodesic flow.

Examples:

The family of deformed spheres – algebraic surfaces of degree 4:

$$x_1^2 + x_2^2 + x_3^2 - 1 + \varepsilon\,\psi(\vec{x}) = 0, \qquad \varepsilon \ll 1,$$

$$\psi(\vec{x}) = \varepsilon_1 x_1^4 + \varepsilon_2 x_2^4 + \varepsilon_3 x_3^4.$$

The study of geodesics on algebraic surfaces is important for applications and for establishing conections of Riemannian geometry to algebraic geometry.

The averaged system:

For the above surface the averaged Hamiltonian is:

$$H = \frac{3}{8}\varepsilon\left[\varepsilon_1\,(L_2^2 + L_3^2)^2 + \varepsilon_2\,(L_1^2 + L_3^2)^2 + \varepsilon_3\,(L_1^2 + L_2^2)^2\right].$$

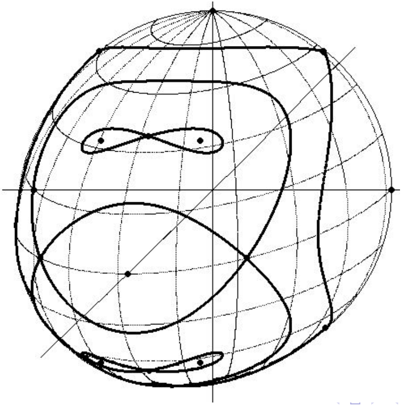
Here L_i are the components of the 3D angular momentum vector.
The Poisson brackets are:

$$\{L_i, L_j\} = \varepsilon_{ijk} L_k.$$

Casimir function: $\vec{L}^2 = L_1^2 + L_2^2 + L_3^2$.
The system can be confined to the sphere $\vec{L}^2 = 1$.

Properties:

Here the averaged system is an integrable system with one degree of freedom. Its trajectories are the level lines of the Hamiltonian $H(\vec{L})$ on the sphere $\vec{L}^2 = 1$.



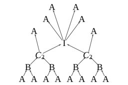
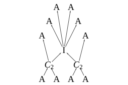
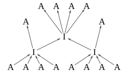
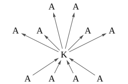
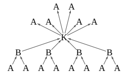
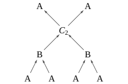
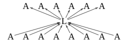
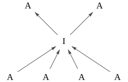
Topological classification:

The topology of the foliation of the phase space into trajectories can be described in terms of the A.T. Fomenko invariant called a molecule. It is an invariant of the foliation with respect to diffeomorphisms of the phase space transforming the level lines into level lines.

The “atoms” A, B, C, etc. correspond to critical levels of the Hamiltonian. They are joined by lines corresponding to sets of non-critical level lines.

Topology in our case:

In our averaged system for the above 4th degree surface, the A.T. Fomenko invariant can be one of the following 8 molecules, depending on the deformation parameters ε_i:

Molecule W_1 ε_i have the same sign, are different, $\varepsilon_1\varepsilon_2-\varepsilon_2\varepsilon_3+\varepsilon_3\varepsilon_1>0$ $\varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3-\varepsilon_3\varepsilon_1>0$ $-\varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3+\varepsilon_3\varepsilon_1>0$		Molecule W_5 ε_i have the same sign, are different, one of the inequalities is false: $\varepsilon_1\varepsilon_2-\varepsilon_2\varepsilon_3+\varepsilon_3\varepsilon_1>0$ $\varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3-\varepsilon_3\varepsilon_1>0$ $-\varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3+\varepsilon_3\varepsilon_1>0$	
Molecule W_2 ε_i have the same sign, the two of ε_i with smaller moduli are equal, $\varepsilon_1\varepsilon_2-\varepsilon_2\varepsilon_3+\varepsilon_3\varepsilon_1>0$ $\varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3-\varepsilon_3\varepsilon_1>0$ $-\varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3+\varepsilon_3\varepsilon_1>0$		Molecule W_6 ε_i have the same sign, two of them are equal, one of the inequalities is false: $\varepsilon_1\varepsilon_2-\varepsilon_2\varepsilon_3+\varepsilon_3\varepsilon_1>0$ $\varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3-\varepsilon_3\varepsilon_1>0$ $-\varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3+\varepsilon_3\varepsilon_1>0$	
Molecule W_3 ε_i have the same sign, the two of ε_i with larger moduli are equal, $\varepsilon_1\varepsilon_2-\varepsilon_2\varepsilon_3+\varepsilon_3\varepsilon_1>0$ $\varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3-\varepsilon_3\varepsilon_1>0$ $-\varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3+\varepsilon_3\varepsilon_1>0$		Molecule W_7 two of ε_i have the same sign and are different, the third has the opposite sign	
Molecule W_4 all ε_i are equal		Molecule W_8 two of ε_i are equal, the third has the opposite sign	

Considering spheres of higher dimension:

3-spheres.

The averaged system is a Hamiltonian system on so(4) - two degrees of freedom.

For integrability we need one additional integral.

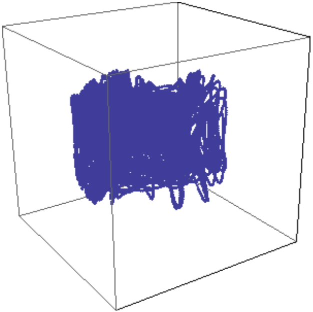
The case of the surfaces of revolution:

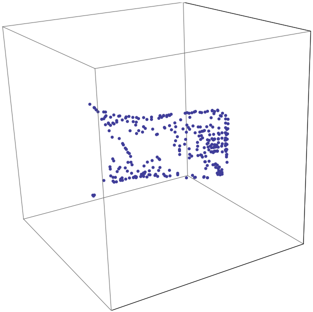
$$\psi(\vec{x}) = f(x_1, x_2, x_3^2 + x_4^2),$$

The averaged system for the 3-sphere with the above deformation is an integrable Hamiltonian system with two degrees of freedom, with the integral I_{34}.

No symmetry - possibility of chaotic dynamics:

The trajectory of the averaged system for S^3 with $\psi(\vec{x}) = \varepsilon_1 x_1^4 + \varepsilon_2 x_2^4 + \varepsilon_3 x_3^4 + \varepsilon_4 x_4^4$ with all ε_i nonzero, and a plane section of the trajectory:





The multidimensional case:

$(n-1)$-dimensional ellipsoid close to the sphere:

$$\sum_{i=1}^{n}(1+\varepsilon\alpha_i)x_i^2 - 1 = 0$$

Theorem. The averaged system for the $(n-1)$-ellipsoid has the Hamiltonian

$$H = \frac{1}{2}\,\varepsilon\sum_{i<j}(\alpha_i + \alpha_j)\, l_{ij}^2 \qquad (1)$$

and coincides with the Manakov integrable case for the Euler equations on the Lie algebra $so(n)$:

$$H = \sum_{i<j}\frac{a_i - a_j}{b_i - b_j}\, l_{ij}^2$$

with $a_i = \varepsilon\alpha_i^2,\ b_i = 2\alpha_i$.

Conclusions:

1. A method for performing an asymptotic reduction of equations for geodesics on weakly deformed spheres using the transformations of integral geometry has been developed.

2. The method allows one to deduce important properties of the averaged systems from theorems of integral geometry.

3. Using the method, the averaged systems for concrete classes of surfaces can be studies, such as the case of 4th degree surfaces.

4. The set of problems arising here calls for a study of Hamiltonian systems on so(n) of arbitrary degree.

5. Asymptotic isomorphisms can provide new information about the relations between the structure of Hamiltonian systems of different dimensions.

Publications:

D. O. Sinitsyn, "Asymptotic Hamiltonian Reduction for Geodesics on Deformed Spheres and the Funk–Minkowski Transform", Mat. Zametki, 90:3 (2011), 474–477.

V. L. Golo and D. O. Sinitsyn, Asymptotic Hamiltonian Reduction for the Dynamics of a Particle on a Surface, Physics of Particles and Nuclei Letters, 2008, Vol. 5, No. 3, pp. 278–281.

arXiv:1003.5449.