Lie-Poisson pencils related to semisimple Lie agebras: towards classification

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Main definition

Definition

A *bi-Lie structure* is a triple $(\mathfrak{g}, [,], [,]')$, where $\mathfrak{g}$ is a vector space and $[,], [,]'$ are two Lie brackets on $\mathfrak{g}$ which are *compatible*, i.e. so that $[,] + [,]'$ is a Lie bracket.

Example

Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{K})$, $A \in \mathfrak{g}$ be a fixed matrix. Put

$$[x,_A y] = xAy - yAx.$$

Then $(\mathfrak{g}, [,], [,_A])$ is a bi-Lie structure, ($[,]$ the standard commutator).

Main motivating example

Let $\mathfrak{g} = \mathfrak{so}(n, \mathbb{K})$, $A \in \mathrm{Symm}(n, \mathbb{K})$, a fixed symmetric matrix. Then $(\mathfrak{g}, [,], [,_A])$ is a bi-Lie structure.

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Motivation I: bihamiltonian structures

Definition

A *bihamiltonian* structure on a manifold M is a pair $\eta_1, \eta_2 \in \Gamma(\bigwedge^2 TM)$ such that $\eta_1, \eta_2, \eta_1 + \eta_2$ are Poisson.

Hierarchy of mechanisms (by complexity of structures):

- constant+constant (rather not interesting)
- constant+linear (proved to be powerful, eg. "argument translation")
- linear+linear (topic of present talk)
- linear+quadratic (eg. argument translation of quadratic bracket towards "vanishing direction")
- etc.

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Semisimple case

Applications of the $\mathfrak{so}(n, \mathbb{R})$ bi-Lie structure:

- Manakov top (n-dimensional free rigid body), here A is diagonal, the "inertia tensor" of the body (Bolsinov 1992)
- Klebsh–Perelomov case (Bolsinov 1992)

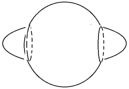
Another bi-Lie structure on $\mathfrak{so}(n, \mathbb{R}) \times \mathfrak{so}(n, \mathbb{R})$

- Generalized Steklov–Lyapunov systems (Bolsinov–Fedorov 1992)

Nonsemisimple case

Works of Golubchik, Odesskii, Sokolov $\sim$ 2004–2006

- Matrix integrable ODE's



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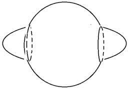

Motivation II: classical R-matrix formalism

Quasigraded Lie algebras

A Lie algebra $(\tilde{\mathfrak{g}}, [,])$ with a decomposition $\tilde{\mathfrak{g}} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ is *quasigraded of degree 1* if $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \oplus \mathfrak{g}_{i+j+1}$

Quasigraded Lie algebras $\to$ standard classical R-matrix

One checks that $\mathfrak{g}_+ := \bigoplus_{n\geq 0} \mathfrak{g}_n, \mathfrak{g}_- := \bigoplus_{n<0} \mathfrak{g}_n$ are subalgebras.

Bi-Lie structures $\to$ quasigraded Lie algebras

Let $(\mathfrak{g}, [,]_0, [,]_1)$ be a bi-Lie structure, $\tilde{\mathfrak{g}} := \mathfrak{g}[\lambda, 1/\lambda]$. Put $[,] = [,]_0 + \lambda[,]_1$ and extend this bracket to $\tilde{\mathfrak{g}}$. Then $\tilde{\mathfrak{g}}$ is quasigraded of degree 1.

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Applications

- Landau-Livshits PDE (the $\mathfrak{so}(n, \mathbb{R})$ bi-Lie structure, $n = 3$, Holod 1987)
- Other finite- and infinite-dimensional systems (Skrypnyk, Golubchik–Sokolov, Yanovski)

Known classification results: the Kantor–Persits theorem

Useful notation

Let $\mathfrak{g}$ be a Lie algebra and $N : \mathfrak{g} \to \mathfrak{g}$ a linear operator. Put

$$[x, y]_N := [Nx, y] + [x, Ny] - N[x, y].$$

Definition

Let $\{[,]^v\}_{v \in V}$ be a n-dimensional vector space of Lie structures on a vector space $\mathfrak{g}$. It is called *irreducible* if the Lie algebras $(\mathfrak{g}, [,]^v)$ do not have common nontrivial ideals and *closed* if

$$\forall x \in \mathfrak{g}\ \forall v, w \in V\ \exists u \in V : [,]^v_{\mathrm{ad}^w x} := [,]^u, \mathrm{ad}^w x(y) = [x, y]^w.$$

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Known classification results: the Kantor–Persits theorem

Kantor–Persits 1988 (announced only)

The list of irreducible closed vector spaces of Lie structures:

- $\mathfrak{g} = \mathfrak{so}(n, \mathbb{K}), \{[,_A]\}_{A \in \mathrm{Symm}(n,\mathbb{K})}$
- $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{K}), \{[,_A]\}_{A \in \mathfrak{m}(n,\mathbb{K})}$
- several *nonsemisimple* cases

here

$$[X,_A Y] := XAY - YAX,$$

$\mathfrak{sp}(n, \mathbb{K}) = \{X \in \mathfrak{gl}(2n, \mathbb{K}) \mid XJ + JX^T = 0\}$ the symplectic Lie algebra, $\mathfrak{m}(n, \mathbb{K}) := \{X \in \mathfrak{gl}(2n, \mathbb{K}) \mid XJ - JX^T = 0\}$ its orthogonal complement in $\mathfrak{gl}(2n, \mathbb{K})$ w.r.t. "trace form"

Odesskii–Sokolov 2006

Classification of "bi-associative structures" $(\cdot, \circ)$ on $\mathfrak{gl}(n, \mathbb{K})$ $\Longrightarrow$ Examples of bi-Lie structures on $\mathfrak{gl}(n, \mathbb{K})$ (which do not restrict to $\mathfrak{sl}(n, \mathbb{K})$)

Semisimple bi-Lie structures and their examples

Definition

Say that a bi-Lie structure $\mathcal{B} := (\mathfrak{g}, [,], [,]')$ is *semisimple* if $(\mathfrak{g}, [,])$ is semisimple.

Known examples of semisimple bi-Lie structures

KP1 $(\mathfrak{so}(n, \mathbb{C}), [,], [,_A])$ (Kantor–Persits 1988)

KP2 $(\mathfrak{sp}(n, \mathbb{C}), [,], [,_A])$ (Kantor–Persits 1988)

GS1 Let $(\mathfrak{g}, [,])$ be semisimple. There exists a bi-Lie structure related to any $\mathbb{Z}_n$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{n-1}$ on $(\mathfrak{g}, [,])$ and to decomposition of the subalgebra $\mathfrak{g}_0 = \mathfrak{g}_0^1 \oplus \mathfrak{g}_0^2$ to two subalgebras (Golubchik–Sokolov 2002)

P Let $(\mathfrak{g}, [,])$ be semisimple. There exists a bi-Lie structure related to any parabolic subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ (P 2006)

GS2 Examples on $\mathfrak{sl}(3, \mathbb{C}), \mathfrak{so}(4, \mathbb{C})$ related to $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings (Golubchik–Sokolov 2002)

Semisimple bi-Lie structures and operators

Obvious or Easy:

Let $(\mathfrak{g}, [,])$ *be a Lie algebra,* $[,]'$ *a bilinear bracket.*

- $[,]'$ *"compatible" with* $[,] \iff [,]'$ *is a 2-cocycle on* $(\mathfrak{g}, [,])$
- *In particular, if* $(\mathfrak{g}, [,], [,]')$ *is a semisimple bi-Lie str., then* $[,]' = [,]_W = [W\cdot, \cdot] + [\cdot, W\cdot] - W[\cdot, \cdot]$ *for some* $W : \mathfrak{g} \to \mathfrak{g}$
- *(Magri–Kosmann-Schwarzbach)* $[,]_N$ *is a Lie bracket for some* $N : \mathfrak{g} \to \mathfrak{g} \iff T_N(\cdot, \cdot) := [N\cdot, N\cdot] - N[\cdot, \cdot]_N$ *is a 2-cocycle on* $(\mathfrak{g}, [,])$
- *In particular,* $(\mathfrak{g}, [,], [,]')$ *is a semisimple bi-Lie str.* $\iff [,]' = [,]_W$ *and* $T_W(\cdot, \cdot) = [\cdot, \cdot]_P$*, where* $P : \mathfrak{g} \to \mathfrak{g}$ *is another linear operator. Moreover, the operators* W, P *are defined up to adding of inner differentiations* $\mathrm{ad}\, x$*.*

$$T_N(X, Y) = [NX, NY] - N([NX, Y] + [X, NY] - N[X, Y])$$

Semisimple bi-Lie structures: examples of leading operators

Definition

Given a semisimple bi-Lie structure $\mathcal{B}$ call W such that $[,]' = [,]_W$ a *leading operator* for $\mathcal{B}$ and P a *primitive* for W. They satisfy *the main identity (MI)*

$$T_W(\cdot,\cdot) = [\cdot,\cdot]_P$$

Example

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{n-1}$ *be a* $\mathbb{Z}/n\mathbb{Z}$*-grading on* $\mathfrak{g}$. *Put* $W|_{\mathfrak{g}_i} = i\mathrm{Id}_{\mathfrak{g}_i}, i = 0,\ldots,n-1$ *and* $P|_{\mathfrak{g}_i} = \frac{1}{2}i(n-i)\mathrm{Id}_{\mathfrak{g}_i}$. *One checks MI directly.*

Example

Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ (sum of subalgebras). Put $W|_{\mathfrak{g}_i} = \omega_i\mathrm{Id}_{\mathfrak{g}_i}, i = 1,2$, where $\omega_{1,2}$ are any scalars. Then $T_W = 0$ (so put $P = 0$ in the MI). Important example: $\mathfrak{g}$ simple, $\mathfrak{g}_1$ a parabolic subalgebra and $\mathfrak{g}_2$ its "complement".

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Principal leading operator

Definition

Let $\mathfrak{g}$ be a semisimple Lie algebra. Then there exists a direct decomposition $\mathrm{End}(\mathfrak{g}) = \mathrm{ad}\,\mathfrak{g} \oplus C$, where $C = (\mathrm{ad}\,\mathfrak{g})^{\perp}$ is the direct complement to $\mathrm{ad}\,\mathfrak{g} \subset \mathrm{End}(\mathfrak{g})$ w.r.t. the trace form. An operator $W \in \mathrm{End}(\mathfrak{g})$ is called *principal* if $W \in C$.

Theorem

1. *There exists a unique principal operator W with the property $[,]' = [,]_W$. Call it the* principal *(leading) operator of a bi-Lie structure $(\mathfrak{g}, [,], [,]')$.*
2. *If W is the principal operator, there exists a unique operator P primitive for W which is* symmetric *w.r.t. the trace form on $\mathrm{End}(\mathfrak{g})$.*

Example

For $\mathfrak{so}(n, \mathbb{K})$ bi-Lie structure we have $W = (1/2)(L_A + R_A)$ (operators of left and right multiplication by A).

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Significance of the principal leading operator

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We say that bi-Lie structures $(\mathfrak{g}, [,], [,]')$ and $(\mathfrak{g}, [,], [,]'')$ are *strongly isomorphic* (*isomorphic*) if there exists an automorphism of the Lie algebra $(\mathfrak{g}, [,])$ sending the bracket $[,]'$ to $[,]''$ (to a linear combination $\alpha_1[,] + \alpha_2[,]''$).

Theorem

Let $(\mathfrak{g}, [,], [,]')$ and $(\mathfrak{g}, [,], [,]'')$ be two semisimple bi-Lie structures and let W', W'' be the corresponding principal operators. Then the bi-Lie structures are strongly isomorphic if and only if there exists an automorphism ϕ of the Lie algebra $(\mathfrak{g}, [,])$ with the property $\phi \circ W' = W'' \circ \phi$.

In particular, classification of semisimple bi-Lie structures up to isomorphism $\Longleftrightarrow$ classification of principal operators satisfyting MI up to action of automorphisms, rescaling, and adding scalar operators

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The pencil of Lie algebras and the times

Switch to $\mathbb{K} = \mathbb{C}$

Bi-Lie structure $(\mathfrak{g}, [,], [,]') \Longrightarrow$ Pencil of Lie brackets $(\mathfrak{g}, [,]^t), [,]^t := [,]' - t[,], t \in \mathbb{C}$

Theorem

Let $(\mathfrak{g}, [,], [,]')$ be a semisimple bi-Lie structure, W its principal operator, P its symmetric primitive and let $B(,)$ be the Killing form of $(\mathfrak{g}, [,])$. Then the Killing form B^t of the Lie algebra $(\mathfrak{g}, [,]^t)$ is given by the formula

$$B^t(x, y) = B((W - tI)x, (W - tI)y) - 2B(Px, y), \ x, y \in \mathfrak{g},$$

In particular, $\ker B^t \neq \{0\} \Longleftrightarrow \det(W^*W - 2P - t(W + W^*) + t^2 I) = 0.$

Definition

The elements of the finite set $T := \{t \in \mathbb{C} \mid \ker B^t \neq \{0\}\}$ are called the *times* of the bi-Lie structure.

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The central subalgebra

In particular, if $t \in T$, the center $\mathfrak{z}^t$ of the Lie algebra $(\mathfrak{g}, [,]^t)$ can be nontrivial. Put $\Theta := \{t \in T \mid \mathfrak{z}^t \neq \{0\}\}$.

Theorem

For $x \in \mathfrak{g}, x \neq 0$, the following conditions are equivalent:

1. *x belongs to the center $\mathfrak{z}^\theta$ of the bracket $[,]^\theta := [,]' - \theta[,] = [,]_{W^\theta}$, here $W^\theta := W - \theta I$;*
2. *x is the eigenvector of the principal operator W corresponding to the eigenvalue θ and $[\mathrm{ad}\, x, W] = 0$.*

Theorem

1. *The subset $\mathfrak{z}^\theta$ is a subalgebra in $(\mathfrak{g}, [,])$ for any $\theta \in \Theta$;*
2. *$\mathfrak{z}^{\theta_i} \cap \mathfrak{z}^{\theta_j} = \{0\}$ if $\theta_i \neq \theta_j$;*
3. *$[\mathfrak{z}^{\theta_i}, \mathfrak{z}^{\theta_j}] = 0$ if $\theta_i \neq \theta_j$; in particular, the set $\mathfrak{z} := \mathfrak{z}^{\theta_1} \oplus \cdots \oplus \mathfrak{z}^{\theta_m}$ is a subalgebra in $(\mathfrak{g}, [,])$ which is a direct sum of its ideals $\mathfrak{z}^{\theta_i}$. Call $\mathfrak{z}$ the* central subalgebra *of $(\mathfrak{g}, [,], [,]')$. Moreover, $\mathfrak{z} \subset \ker P$.*

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2. *$\mathfrak{z}^{\theta_i} \cap \mathfrak{z}^{\theta_j} = \{0\}$ if $\theta_i \neq \theta_j$;*
3. *$[\mathfrak{z}^{\theta_i}, \mathfrak{z}^{\theta_j}] = 0$ if $\theta_i \neq \theta_j$; in particular, the set $\mathfrak{z} := \mathfrak{z}^{\theta_1} \oplus \cdots \oplus \mathfrak{z}^{\theta_m}$ is a subalgebra in $(\mathfrak{g}, [,])$ which is a direct sum of its ideals $\mathfrak{z}^{\theta_i}$. Call $\mathfrak{z}$ the* central subalgebra *of $(\mathfrak{g}, [,], [,]')$. Moreover, $\mathfrak{z} \subset \ker P$.*

Examples of the central subalgebras

$(\mathfrak{so}(6,\mathbb{C}),[,],[,_A])$

$$A = \begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & c \end{bmatrix}, \mathfrak{z} = \begin{bmatrix} * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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i-Lie $\leftrightsquigarrow$ $\mathbb{Z}/n\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{n-1} \Longrightarrow \mathfrak{z} = \mathfrak{g}_0$.

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Gradings and Main assumption

Definition

Let $\mathfrak{g} = \bigoplus_{i \in \Gamma} \mathfrak{g}_i$ be a *grading* of a Lie algebra $(\mathfrak{g}, [,])$, i.e. $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for any $i, j \in \Gamma$, Γ an abelian group. We say that a linear operator $W : \mathfrak{g} \to \mathfrak{g}$ *preserves* the grading if $W\mathfrak{g}_i \subset \mathfrak{g}_i$ for any $i \in \Gamma$.

Theorem

Let $(\mathfrak{g}, [,], [,]')$ be a semisimple bi-Lie structure and let $\mathfrak{g} = \bigoplus_{i \in \Gamma} \mathfrak{g}_i$ be a grading. Then, if the principal operator $W : \mathfrak{g} \to \mathfrak{g}$ preserves the grading, so does its symmetric primitive P.

Main assumption: $\mathfrak{z} \supset \mathfrak{h}$

The central subalgebra $\mathfrak{z}$ contains some Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ (w.r.t.$[,]$). *one slide back*

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Gradings and Main assumption

Theorem

The main assumption $\mathfrak{z} \supset \mathfrak{h}$ is equivalent to the following two conditions

- The principal operator $W \in \mathrm{End}(\mathfrak{g})$ preserves the grading

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in R} \mathfrak{g}_\alpha$$

  related to the root decomposition with respect to the Cartan subalgebra $\mathfrak{h}$. In other words for some $\omega_\alpha \in \mathbb{C}$

$$W|_{\mathfrak{g}_\alpha} = \omega_\alpha \mathrm{Id}_{\mathfrak{g}_\alpha}, W\mathfrak{h} \subset \mathfrak{h}.$$

- The operator $W|_{\mathfrak{h}}$ is diagonalizable.

Consequences of the Main assumption

Theorem

Recall $W|_{\mathfrak{g}_\alpha} = \omega_\alpha \mathrm{Id}_{\mathfrak{g}_\alpha}, W\mathfrak{h} \subset \mathfrak{h}, P|_{\mathfrak{g}_\alpha} = \pi_\alpha \mathrm{Id}_{\mathfrak{g}_\alpha}, \pi_\alpha = \pi_{-\alpha},$ *put* $P|_{\mathfrak{h}} = 0.$

Then for any $\alpha \in R$

- *there exist two* times $t_{1,\alpha}, t_{2,\alpha}$ *(possibly equal) such that* $\mathfrak{g}_\alpha \subset \ker B^{t_{1,\alpha}} \cap \ker B^{t_{2,\alpha}}$. *They are the solutions of the quadratic equation* $(t-\omega_\alpha)(t-\omega_{-\alpha}) - 2\pi_\alpha = 0$. *Moreover, if* $T_\alpha := \{t_{1,\alpha}, t_{2,\alpha}\}$, *then* $T_\alpha = T_{-\alpha}$.
- $\sigma_\alpha = (1/2)(t_{1,\alpha} + t_{2,\alpha}), \kappa_\alpha = \pm\sqrt{((t_{1,\alpha} - t_{2,\alpha})/2)^2 - 2\pi_\alpha}$, *where* $\sigma_\alpha := (1/2)(\omega_\alpha + \omega_{-\alpha}), \kappa_\alpha := (1/2)(\omega_\alpha - \omega_{-\alpha})$.
- $(W - t_{1,\alpha} I)(W - t_{2,\alpha} I)H_\alpha = 0$, *here* $H_\alpha \in \mathfrak{h}, \alpha \in R$, *is such that* $B(H_\alpha, H) = \alpha(H)$ *for any* $H \in \mathfrak{h}$. *Consequently,* H_α *is either an eigenvector of* W *corresponding to the eigenvalue* $t_{1,\alpha}$, *or an eigenvector of* W *corresponding to the eigenvalue* $t_{2,\alpha}$, *or a sum of such eigenvectors. Hence* $W|_{\mathfrak{h}}$ *is* admissible *in the following sense:*

Admissible operators

Definition

Let V be a vector space over $\mathbb{R}$ and let $R \subset V$ be a reduced root system in V. A diagonalizable linear operator $U : V^{\mathbb{C}} \to V^{\mathbb{C}}$ will be called R-*admissible* if for any $\alpha \in R \subset V^{\mathbb{C}}$

1. either there exist two eigenvectors $w_{1,\alpha}, w_{2,\alpha} \in V^{\mathbb{C}}$ corresponding to different eigenvalues $t_{1,\alpha}, t_{2,\alpha}$ of the operator U such that

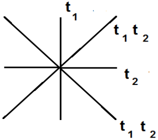
$$\alpha = w_{1,\alpha} + w_{2,\alpha};$$

2. or α is an eigenvector of U corresponding to the eigenvalue t_α.

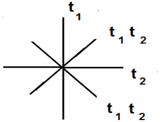
Put $U_\alpha := \{t_{1,\alpha}, t_{2,\alpha}\}$ or $U_\alpha := \{t_\alpha\}$.

Examples of admissible operators

- Any diagonalizable operator with two eigenvalues is admissible.
- $R = \mathfrak{d}_n$, roots $\pm\epsilon_i \pm \epsilon_j (1 \le i < j \le n)$, here ϵ_i elements of the standard basis in $\mathbb{R}^n$. Put $U\epsilon_i = t_i\epsilon_i$ (KP1 with $A = \mathrm{diag}(t_1, t_1, \ldots, t_n, t_n)$).
- $R = \mathfrak{b}_n$, roots $\pm\epsilon_i (1 \le i \le n), \pm\epsilon_i \pm \epsilon_j (1 \le i < j \le n)$. Again $U\epsilon_i = t_i\epsilon_i$ (KP1 with $A = \mathrm{diag}(t_1, t_1, \ldots, t_n, t_n, t_{n+1})$).



- $R = \mathfrak{c}_n$, roots $\pm 2\epsilon_i (1 \le i \le n), \pm\epsilon_i \pm \epsilon_j (1 \le i < j \le n)$. Again $U\epsilon_i = t_i\epsilon_i$ (KP2 with $A = \mathrm{diag}(t_1, t_1, \ldots, t_n, t_n)$).



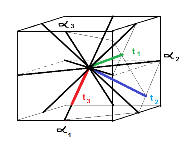
Examples of admissible operators

Example

$R = \mathfrak{a}_n$, root basis $\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \alpha_n = \epsilon_n - \epsilon_{n+1}$. Put

$$\begin{array}{lclclclcl}
w_n & := & a\alpha_n & & & & & & \\
w_{n-1} & := & w_n & + & \alpha_{n-1} & & & & \\
w_{n-2} & := & & & w_{n-1} & + & \alpha_{n-2} & & \\
 & \vdots & & & & & & & \\
w_1 & := & & & & & w_2 & + & \alpha_1,
\end{array}$$

where $a \neq 0, 1$ is a complex parameter, and $U(w_i) := t_i w_i$. Then $\alpha_n = (1/a)w_n, \alpha_{n-1} = w_{n-1} - w_n, \alpha_{n-2} = w_{n-2} - w_{n-1}, \ldots, \alpha_1 = w_1 - w_2$.

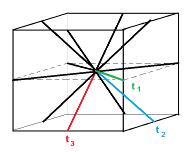


$$\begin{array}{ccccc}
 & & t_1t_3 & & \\
 & t_1t_3 & & t_2t_3 & \\
t_1t_2 & & t_2t_3 & & t_3
\end{array}$$

Examples of admissible operators

Example

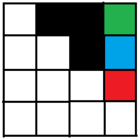
$R = \mathfrak{a}_n$. Put $a = 1$ in previous example.



t_1

$t_1 t_3$ t_2

$t_1 t_2$ $t_2 t_3$ t_3



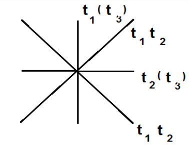
Virtual times

Definition

If $\alpha \in R$ is an eigenvector for $W|_{\mathfrak{h}}$ corresponding to eigenvalue $t_{1,\alpha}$ and $T_\alpha = \{t_{1,\alpha}, t_{2,\alpha}\}$, $t_{2,\alpha} \neq t_{1,\alpha}$, we call $t_{2,\alpha}$ a *virtual time*

Example: KP1 on $\mathfrak{so}(5)$ with $A = \mathrm{diag}(t_1, t_1, t_2, t_2, t_3)$

$$\begin{bmatrix} 0 & 2t_1 & t_1+t_2 & t_1+t_2 & t_1+t_3 \\ 2t_1 & 0 & t_1+t_2 & t_1+t_2 & t_1+t_3 \\ t_1+t_2 & t_1+t_2 & 0 & 2t_2 & t_2+t_3 \\ t_1+t_2 & t_1+t_2 & 2t_2 & 0 & t_2+t_3 \\ t_1+t_3 & t_1+t_3 & t_2+t_3 & t_2+t_3 & 0 \end{bmatrix}$$



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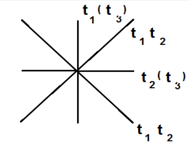

"Times selection rules" and antisymmetric part of W

Theorem

Let $\alpha, \beta, \gamma \in R$ be such that $\alpha + \beta + \gamma = 0$. Then only the following possibilities can occur:

1. *either there exist $t_1, t_2, t_3 \in \mathbb{C}$ such that*
$$T_\alpha = \{t_1, t_2\},\ T_\beta = \{t_2, t_3\},\ T_\gamma = \{t_3, t_1\};$$
2. *or there exist $t_1, t_2 \in \mathbb{C}$ such that*
$$T_\alpha = T_\beta = T_\gamma = \{t_1, t_2\}, t_1 \neq t_2,$$

Moreover, in Case 1 the following equality holds:

$$\kappa_\alpha + \kappa_\beta + \kappa_\gamma = 0$$

and in Case 2:

$$\kappa_\alpha + \kappa_\beta + \kappa_\gamma = \pm(t_1 - t_2)/2.$$

Pairs diagrams

Definition

Let R be a reduced root system. A collection $\{T_\alpha\}_{\alpha\in R}$ of unordered pairs $T_\alpha = \{t_{1,\alpha}, t_{2,\alpha}\}$ of complex numbers is called a *pairs diagram* if

1. $T_\alpha = T_{-\alpha}$ for any $\alpha \in R$;
2. for any $\alpha, \beta, \gamma \in R$ such that $\alpha + \beta + \gamma = 0$ the pairs $T_\alpha, T_\beta, T_\gamma$ obey "the times selection rules".

A pairs diagram $\{T_\alpha\}_{\alpha\in R}$ is called *admissible* if there exists an admissible operator $U : V^{\mathbb{C}} \to V^{\mathbb{C}}$ such that $U_\alpha \subset T_\alpha$ for any $\alpha \in R$; the pair $(U, \{T_\alpha\}_{\alpha\in R})$ will be called *admissible* too.

Examples:

t_1t_3

t_1t_2 $\quad$ t_2t_3 ,

$t_1(t_4)$

t_1t_3 $\quad$ $t_2(t_4)$

t_1t_2 $\quad$ t_2t_3 $\quad$ $t_3(t_4)$,

---

t_1t_3

t_1t_3 $\quad$ t_2t_3

t_1t_2 $\quad$ t_2t_3 $\quad$ t_3t_3 ,

t_1t_2

t_1t_2 $\quad$ t_1t_2

t_1t_1 $\quad$ t_1t_2 $\quad$ t_1t_2 .

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$$\begin{array}{ccc} & t_1t_3 & \\ t_1t_2 & & t_2t_3 \end{array}, \qquad \begin{array}{ccccc} & & t_1(t_4) & & \\ & t_1t_3 & & t_2(t_4) & \\ t_1t_2 & & t_2t_3 & & t_3(t_4) \end{array},$$

$$\begin{array}{ccccc} & & t_1t_3 & & \\ & t_1t_3 & & t_2t_3 & \\ t_1t_2 & & t_2t_3 & & t_3t_3 \end{array}, \qquad \begin{array}{ccccc} & & t_1t_2 & & \\ & t_1t_2 & & t_1t_2 & \\ t_1t_1 & & t_1t_2 & & t_1t_2 \end{array}.$$

Two classes of pairs diagrams

Theorem

Let R be a reduced irreducible root system and let $\{T_\alpha\}_{\alpha\in R}$ be a pairs diagram. Assume that there exist $\alpha, \beta, \gamma \in R$ such that $\alpha+\beta+\gamma = 0$ and

$$T_\alpha = T_\beta = T_\gamma = \{t_1, t_2\}$$

for some $t_1, t_2 \in \mathbb{C}, t_1 \neq t_2$. Then $T_\delta = \{t_1, t_2\}, \{t_1, t_1\}$ or $\{t_2, t_2\}$ for any $\delta \in R$.

Definition

We say that a pairs diagram $\{T_\alpha\}_{\alpha\in R}$ is of *Class II*, if there exist $\alpha, \beta, \gamma \in R$ satisfying the hypotheses of the theorem, and of *Class I*, if such roots do not exist.
A bi-Lie structure is of *Class I* or *II* correspondingly to the class of diagram.

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Bi-Lie structures of Class I

Theorem

Given an admissible pair $(U, \mathcal{T}), \mathcal{T} := \{T_\alpha\}_{\alpha \in R}$, where $\mathcal{T}$ is of Class I,

- *there exists a unique operator $W : \mathfrak{g} \to \mathfrak{g}$ such that $W|_{\mathfrak{h}} = U$ and W is a principal leading operator for a bi-Lie structure.*
- *It is of the form $W|_{\mathfrak{g}_\alpha + \mathfrak{a}_{-\alpha}} = [(t_{1,\alpha} + t_{2,\alpha})/2]\mathrm{Id}_{\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}}$ and is symmetric iff so is U.*
- *The central subalgebra $\mathfrak{z}$ consists of $\mathfrak{h}$ and those $\mathfrak{g}_\alpha$ for which $T_\alpha = \{t_i, t_i\}$ for some time t_i and the sum of eigenspaces of U corresponding to eigenvalues $t \neq t_i$ is orthogonal to the root α.*

Theorem

Each pairs diagram of Class I induces a specific $\mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z}$-grading on the Lie algebra $(\mathfrak{g}, [,])$.

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Examples of admissible pairs of Class I

Example

Let R be arbitrary, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ any $\mathbb{Z}_2$-grading induced by an *inner* automorphism of order 2 and let $\mathfrak{g}_0 = \mathfrak{g}_0^1 \oplus \mathfrak{g}_0^2$ be a decomposition of $\mathfrak{g}_0$ to subalgebras. Then $\mathfrak{g}_0^i = \mathfrak{h} \cap \mathfrak{g}_0^i + \sum_{\alpha \in R^i} \mathfrak{g}_\alpha$, where $R^i \subset R$ is a closed symmetric root subsystem. Put $T_\alpha := \{t_i, t_i\}$ for $\alpha \in R^i, i = 1, 2$, $T_\alpha := \{t_1, t_2\}$ for $\alpha \in R \setminus (R^1 \cup R^2)$ and $U|_{\mathfrak{h} \cap \mathfrak{g}_0^i} = t_i \mathrm{Id}_{\mathfrak{h} \cap \mathfrak{g}_0^i}$ (GS1 with $\mathbb{Z}_2$-grading related to an *inner* involutive automorphism).

$$\begin{array}{ccccc} & & t_1 t_2 & & \\ & t_1 t_2 & & t_1 t_2 & \\ t_1 t_1 & & t_1 t_2 & & t_2 t_2 \end{array}$$

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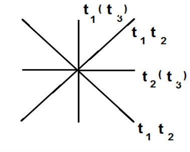
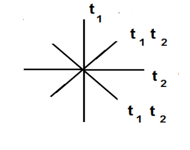
$R = \mathfrak{d}_n$, roots $\pm\epsilon_i \pm \epsilon_j (1 \le i < j \le n)$, $U\epsilon_i = t_i\epsilon_i$, $T_{\pm\epsilon_i\pm\epsilon_j} := \{t_i, t_j\}$ (KP1, $A = \mathrm{diag}(t_1, t_1, \dots, t_n, t_n)$).

Example

$R = \mathfrak{b}_n$, *roots* $\pm\epsilon_i (1 \le i \le n), \pm\epsilon_i \pm \epsilon_j (1 \le i < j \le n) U\epsilon_i = t_i\epsilon_i$, $T_{\pm\epsilon_i\pm\epsilon_j} := \{t_i, t_j\}$, $T_{\pm\epsilon_i} := \{t_i, (t_{n+1})\}$ *(KP1, $A = \mathrm{diag}(t_1, t_1, \dots, t_n, t_n, t_{n+1})$).*



$R = \mathfrak{c}_n$, *roots* $\pm 2\epsilon_i (1 \le i \le n), \pm\epsilon_i \pm \epsilon_j (1 \le i < j \le n) U\epsilon_i = t_i\epsilon_i$, $T_{\pm\epsilon_i\pm\epsilon_j} := \{t_i, t_j\}$, $T_{\pm 2\epsilon_i} := \{t_i, t_i\}$ *(KP2, $A = \mathrm{diag}(t_1, t_1, \dots, t_n, t_n)$).*



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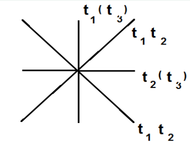
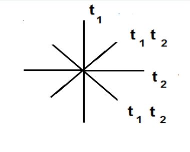
Example

$R = \mathfrak{b}_n$, roots $\pm\epsilon_i (1 \le i \le n), \pm\epsilon_i \pm \epsilon_j (1 \le i < j \le n) U\epsilon_i = t_i\epsilon_i$, $T_{\pm\epsilon_i \pm \epsilon_j} := \{t_i, t_j\}$, $T_{\pm\epsilon_i} := \{t_i, (t_{n+1})\}$ *(KP1,* $A = \mathrm{diag}(t_1, t_1, \ldots, t_n, t_n, t_{n+1})$*).*



$R = \mathfrak{c}_n$, roots $\pm 2\epsilon_i (1 \le i \le n), \pm\epsilon_i \pm \epsilon_j (1 \le i < j \le n) U\epsilon_i = t_i\epsilon_i$, $T_{\pm\epsilon_i \pm \epsilon_j} := \{t_i, t_j\}$, $T_{\pm 2\epsilon_i} := \{t_i, t_i\}$ *(KP2,*$A = \mathrm{diag}(t_1, t_1, \ldots, t_n, t_n)$*).*



Examples of admissible pairs of Class I

$R = \mathfrak{a}_n$, root basis $\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \alpha_n = \epsilon_n - \epsilon_{n+1}$.

a) Put $w_n := a\alpha_n, w_{n-1} := w_n + \alpha_{n-1}, \ldots, w_1 := w_2 + \alpha_1$, where $a \neq 0, 1$, $U(w_i) := t_i w_i$, $T_{\pm(\epsilon_i - \epsilon_j)} := \{t_i t_j\}$, if $i < j < n+1$ and $T_{\pm(\epsilon_i - \epsilon_{n+1})} = \{t_i t_n\}$ (new).

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b) Put $a = 1$ and $T_{\pm(\epsilon_i - \epsilon_{n+1})} = \{t_i (t_{n+1})\}$

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(new, corresponds to $WX = (1/2)(L_A + R_A)X - \mathrm{Tr}((1/2)(L_A + R_A)X)B$, where $X \in \mathfrak{sl}(n+1), A = \mathrm{diag}(t_1, t_2, \ldots, t_{n+1}), B = \mathrm{diag}(0, 0, \ldots, 0, 1)$).

Conjecture

Any bi-Lie structure of Class I is obtained either from one of the admissible pairs listed or by a reduction (by means of identifying some of the parameters $t_1, \ldots, t_n$).

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Bi-Lie structures of Class II

Theorem

Given an admissible pair $(U, \mathcal{T}), \mathcal{T} := \{T_\alpha\}_{\alpha \in R}$, where $\mathcal{T}$ is of Class II,

$$\begin{array}{ccccccc} & & & t_1 t_2 & & & \\ \textit{eg.} & & t_1 t_2 & & t_1 t_2 & & \\ & t_1 t_1 & & t_1 t_2 & & t_1 t_2, & \end{array}$$

assume that $W : \mathfrak{g} \to \mathfrak{g}$ is an operator such that $W|_{\mathfrak{h}} = U$ and

- *Its symmetric part on $\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}$ is of the form $W|_{\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}} = [(t_{1,\alpha} + t_{2,\alpha})/2]\mathrm{Id}_{\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}}$ (here $\{t_{1,\alpha}, t_{2,\alpha}\} = \{t_1, t_2\}, \{t_1, t_1\}$ or $\{t_2, t_2\}$).*
- *Its antisymmetric part satisfies the "$(t_1 - t_2)/2$-triangle rule".*

Then

- *W is a leading operator for a bi-Lie structure of Class II.*
- *The central subalgebra $\mathfrak{z}$ consists of $\mathfrak{h}$ and those $\mathfrak{g}_\alpha$ for which $T_\alpha = \{t_i, t_i\}$ for some time t_i and the sum of eigenspaces of U corresponding to eigenvalues $t \neq t_i$ is orthogonal to the root α.*

Bi-Lie structures of Class II

x-triangle rule

Let $\alpha, \beta, \gamma \in R$ be such that $\alpha + \beta + \gamma = 0$.

1. if
$$T_\alpha = \{t_1, t_1\},\ T_\beta = \{t_1, t_2\},\ T_\gamma = \{t_2, t_1\}$$
then
$$\kappa_\alpha + \kappa_\beta + \kappa_\gamma = 0.$$
2. if
$$T_\alpha = T_\beta = T_\gamma = \{t_1, t_2\},\ t_1 \neq t_2,$$
then
$$\kappa_\alpha + \kappa_\beta + \kappa_\gamma = \pm x.$$

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Examples of bi-Lie structures of Class II

Example 1

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{n-1}$ be a $\mathbb{Z}_n$-grading on $\mathfrak{g}$ related to an *inner* automorphism of n-th order, $n > 2$, and $W|_{\mathfrak{g}_i} = i\mathrm{Id}_{\mathfrak{g}_i}, i = 0, \ldots, n-1$ (GS1 with *inner* automorphism of n-th order, $n > 2$).

Example 2

Let $\mathfrak{g} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1$, where $\tilde{\mathfrak{g}}_0$ is a parabolic subalgebra and $\tilde{\mathfrak{g}}_1$ its "complement", $W|_{\tilde{\mathfrak{g}}_i} = \omega_i \mathrm{Id}_{\tilde{\mathfrak{g}}_i}$, ω_i arbitrary (P).

Theorem

Any Example 2 is isomorphic to one of the Examples 1 (for which $\mathfrak{g}_0$ is a Levi subalgebra)

$$\mathfrak{sl}(5) : \begin{bmatrix} * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}$$

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Regular reductive subalgebras

A reductive subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ is *regular* if it contains some Cartan subalgebra $\mathfrak{h}$. There is a one-to-one correspondence $\{$Regular reductive subalgebras$\} \leftrightarrow \{$closed symmetric subsytems $R_0 \subset R\}$

Classification of regular reductive subalgebras

$L := \mathrm{Span}_{\mathbb{Z}} R, L_0 := \mathrm{Span}_{\mathbb{Z}} R_0, \Gamma(\mathfrak{g}_0) := L/L_0$

1. $\Gamma(\mathfrak{g}_0)$ is free - Levi subalgebra;
2. The torsion component $\mathrm{Tor}\Gamma(\mathfrak{g}_0)$ is cyclic;
3. $\mathrm{Tor}\Gamma(\mathfrak{g}_0)$ is not cyclic.

New example of bi-Lie structure of Class II

$\mathfrak{g} = \mathfrak{e}_7, \Gamma(\mathfrak{g}_0) = \mathbb{Z}_3 \times \mathbb{Z}_3$ corresponds to case 3, $\mathfrak{g}_0$ is not the fixed point subalgebra of an inner automorphism of finite order (the last corresponds to cases 1 or 2)

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Many thanks!

http://arxiv.org/abs/1208.1642