Tracing KAM tori in presymplectic dynamical systems

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Abstract

We present a KAM theorem for presymplectic dynamical systems. The theorem has a " a posteriori " format. We show that given a Diophantine frequency ω and a family of presymplectic mappings, if we find an embedded torus which is approximately invariant with rotation ω such that the torus and the family of mappings satisfy some explicit non-degeneracy condition, then we can find an embedded torus and a value of the parameter close to to the original ones so that the torus is invariant under the map associated to the value of the parameter. Furthermore, we show that the dimension of the parameter space is reduced if we assume that the systems are exact.

1. Presymplectic Dynamics

Presymplectic structures (constant rank, closed 2-forms) arise naturally in the study of degenerate Lagrangian and Hamiltonian mechanical systems with constrains, in time dependent Hamiltonian systems and in control theory.

Given a presymplectic form $\Omega \in \Omega^2(M)$, a vector field $X \in \mathfrak{X}(M)$ is said to be a Hamiltonian vector field associated with a function $H \in C^\infty(M)$ if:

$$i_X\Omega = \mathrm{d}H.$$

Due to the degeneracy of Ω, there can be different functions H associated with X, not differing by a constant. The corresponding flow $\phi_X^t : M \to M$ is a 1-parameter group of presymplectic diffeomorphisms: $(\phi_X^t)^*\Omega = \Omega$. Hence, the dynamics of such systems leave the presymplectic structure invariant.

EXAMPLE 1.1. *Example to keep in mind could be the three dimensional torus endowed with a presymplectic form $\Omega = d\Psi_1 \wedge d\Psi_2$. Clearly, the kernel is given by the level sets of Ψ_1, Ψ_2.*

EXAMPLE 1.2. *A more complicated example on $\mathbb{T}^3$ is $\Omega = d\Psi_1 \wedge \gamma$ where γ is a closed but not exact form. In this case, the kernel can be an irrational foliation.*

EXAMPLE 1.3. *Another example related to the previous ones is the study of quasi-periodically perturbed Hamiltonian systems $H(x, \omega t)$. These can be made autonomous by adding an extra variable $\theta \in \mathbb{T}^d$ that satisfies $\frac{d}{dt}\theta = \omega$. The phase space is now supplemented by a factor $\mathbb{T}^d$. The symplectic form in the phase space becomes a presymplectic form in the extended phase space having $\mathbb{T}^d$ in the kernel.*

Remark. The paper [2] shows how the Pontriaguin maximum principle for optimal trajectories can be formulated using presymplectic systems. If we consider a mechanical system with KAM tori and subject it to a control indexed by enough parameters, the results in this paper give a condition which ensures that the one adjust parameters to maintain the quasi-periodic motion. It would be interesting to study in detail concrete models, specially because the methods we use here, are well suited for numerical implementations.

Remark. The theory of presymplectic manifolds was developed (e.g. in [5]) to give a geometric framework to the Dirac theory of constrained systems, [3, 4]. There are many physically interesting examples of constrained systems to which the present theory applies.

2. Main Result

Let U_ρ denote the complex strip of width $\rho > 0$:

$$U_\rho = \{\theta \in \mathbb{C}^{d+n}/\mathbb{Z}^{d+n} : |Im(\rho)| \leq \rho\},$$

DEFINITION 2.4. *The space $(\mathcal{P}_\rho, \|.\|_\rho)$ consists of functions $K : U_\rho \to M$ which are one periodic in all their arguments, real analytic on the interior of U_ρ and continuous on the closure of U_ρ. We endow this space with the norm*

$$\|K\|_\rho := \sup_{\theta \in U_\rho} |K(\theta)|,$$

which makes it into a Banach space.

DEFINITION 2.5. *Given $\gamma > 0$ and $\sigma \geq d + n$, we will denote by $D(\gamma, \sigma)$ the set of frequency vectors $\omega \in \mathbb{R}^{d+n}$ satisfying the* ***Diophantine condition****:*

$$|l \cdot \omega - m| \geq \gamma |l|_{\mathbb{Z}}^{-\sigma} \quad \forall l \in \mathbb{Z}^{d+n} \backslash \{0\}, m \in \mathbb{Z} \tag{1}$$

Following result by Rüssmann ([8]) is a main ingredient:

Our main result can be stated as follows. We consider the presymplectic manifold

$$M := T^*\mathbb{T}^d \times \mathbb{T}^n, \tag{2}$$

with an **exact** presymplectic form Ω of rank $2d$, whose kernel coincides with the $\mathbb{T}^n$-direction.

One says that $K : \mathbb{T}^{d+n} \to M$ is an invariant torus of a diffeomorphism $f : M \to M$ with frequency $\omega \in \mathbb{R}^{n+d}$ if:

$$f(K(\theta)) - K(\theta + \omega) = 0, \quad \forall \theta \in \mathbb{T}^{n+d}.$$

When the left hand side is non-zero, but small enough (we will assume all the functions can be extended to U_ρ, then we will use the norm $\|.\|_\rho$ to measure the error), one says that f has an *approximate* invariant torus K with frequency ω. Our main theorem can be stated in rough terms as follows:

THEOREM 2.6. *Let $f_\lambda : M \to M$, where M is as in (2), be an analytic, non-degenerate in the sense to be defined later, $(2d + n)$-parametric family of presymplectic diffeomorphisms such that f_0 has an approximate invariant torus K_0, satisfying a non-degeneracy condition, with frequency ω satisfying a Diophantine condition. Then there exists a diffeomorphism f_{λ_∞} in this family, where λ_∞ is close to 0, which has an invariant torus K_∞ with frequency ω and which is close to the initial torus K_0 with respect to norm $\|.\|_{\rho_\infty}$ where $0 < \rho_\infty < \rho$.*

The precise version of theorem can be find at [1].

3. Sketch of proof:

We will use a modified Newton method of the type introduced by Moser in [6, 7, 9]. The procedure goes as follows. Starting with

$$G(K_0, 0) := f_0(K_0(\theta)) - K_0(\theta + \omega) = e_0(\theta), \tag{3}$$

we look for an approximate solution for the corresponding linearized equation

$$DG(K_0, 0)|_{(\Delta_0(\theta), \varepsilon_0)} := \tag{4}$$
$$\frac{\partial f_\lambda(K_0(\theta))}{\partial \lambda}\bigg|_{\lambda=0} \varepsilon_0 + Df_0(K_0(\theta))\Delta_0(\theta) - \Delta_0(\theta + \omega) = -e_0(\theta).$$

By an approximate solution we mean up to a quadratic error, i.e., a solution $\Delta_0(\theta)$ such that:

$$\|DG(K_0, 0)|_{(\Delta_0(\theta), \varepsilon_0)} + e_0\|_{\rho_0 - \delta_0} \leq c_0 \gamma^{-3} \delta_0^{-(3\sigma+1)} \|e_0\|_{\rho_0}^2$$

where δ_0, c_0 are constants to be determined later.

Having the solution $(\Delta_0(\theta), \varepsilon_0)$ a better approximating torus for the map f_{λ_1}, where $\lambda_1 = \lambda_0 + \varepsilon_0$, is defined as

$$K_1(\theta) = K_0(\theta) + \Delta_0(\theta)$$

and it will be shown that $(K_1(\theta), f_{\lambda_1})$ is a non-degenerate pair. Furthermore, setting

$$e_1(\theta) := f_{\lambda_1}(K_1(\theta)) - K_1(\theta)$$

we find that

$$\|e_1\|_{\rho_0 - \delta_0} \leq c_0 \gamma^{-4} \delta_0^{-4\sigma} \|e_0\|_{\rho_0}^2.$$

In other words, for the new torus the error has decreased quadratically.

Iterating this procedure, we will see that the sequence

$$(K_0, \lambda_0), (K_1, \lambda_1), \ldots, (K_n, \lambda_n), \ldots$$

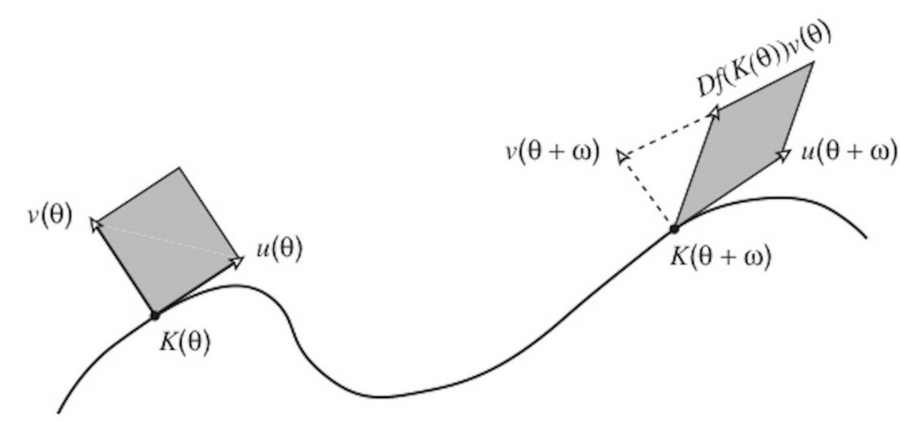
of approximate invariant tori for the functions paired with them, obtained by applying the iterative procedure, converges to a solution $(K_\infty, \lambda_\infty)$. One has to be careful with the domain U_ρ which decreases in each iteration (the reason is because we can bound the correction applied at one step only in a domain slightly smaller than the domain of the original function, see Proposition (4.7)). This loss of domain can be arranged in a way that, in the limit, one does not end up with an empty domain. This choice of decreasing domains so that there is some domain that remains is very standard in KAM theory.

4. Approximate solution:

The non-degeneracy condition we impose and perseverance of the presymplectic structure will provide us a matrix valued map, $M(\theta)$ such that for two matrices $s(\theta), A(\theta)$, the equation

$$Df(K(\theta)) \cdot M(\theta)) - M(\theta + \omega) \cdot \begin{bmatrix} I_d & S_1(\theta) & 0 \\ 0 & I_d & 0 \\ 0 & A(\theta) & I_n \end{bmatrix} = 0. \tag{5}$$

holds up to an error. Following figure gives a vision in dimension two (symplectic case)



The change of variable $\Delta_0(\theta) = M(\theta)\xi(\theta)$ will transform (4) to

$$\left(\begin{bmatrix} I_d & S(\theta) & 0 \\ 0 & I_d & 0 \\ 0 & A(\theta) & I_n \end{bmatrix} \right) \xi(\theta) - \xi(\theta + \omega) = h(\theta), \tag{6}$$

where $h(\theta)$ has terms bounded by quadratic error that we will ignore, since we are looking for an approximate solution. Then we will solve the transformed equation using following proposition:

PROPOSITION 4.7 (Cohomological Equation)**.** *Let $\omega \in D(\sigma, \gamma)$ and assume that $h : \mathbb{T}^{d+n} \to \mathbb{R}^{2d+n}$ is analytic on U_ρ and has zero average,* $\mathrm{avg}(h) := \int_{\mathbb{T}^{d+n}} h(\theta) \mathrm{d}\theta = 0$. *Then for all $0 < \delta < \rho$, the difference equation*

$$v(\theta) - v(\theta + \omega) = h(\theta) \tag{7}$$

has a unique zero average solution $v : \mathbb{T}^{d+n} \to \mathbb{R}^{2d+n}$ which is analytic in $U_{\rho-\delta}$. Moreover, this solution satisfies the following estimate:

$$\|v\|_{\rho-\delta} \leq c_0 \gamma^{-1} \delta^{-\sigma} \|h\|_\rho, \tag{8}$$

where c_0 is a constant depending on n and σ.

Remark. The average of $h(\theta)$ in the right hand side of (6) is not zero in general. An other advantage of the non-degeneracy condition we impose is that we can use parameters to kill the average. As we mentioned before, this non-degeneracy condition will stay holding all over the iteration procedure.

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