A Poisson Zoo of Integrable Systems
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Outline

But what is a Poisson manifold?

Definition

A Poisson structure on a smooth manifold M is given by a smooth bivector field Π satisfying

$$[\Pi, \Pi] = 0$$

This defines a Poisson bracket on $\mathcal{C}^\infty(M)$,

$$\{f, g\} := \Pi(df, dg)$$

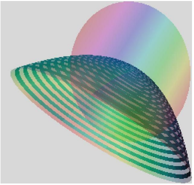
And the manifold M is endowed with a smooth foliation (in the Sussmann sense) whose leaves are symplectic manifolds.

The symplectic foliation is then spanned by the vector fields $\Pi(df, .)$ with $f \in \mathcal{C}^\infty(M)$.

Local Structure. Weinstein's theorem.

The Poisson manifold is locally a product of a symplectic manifold with a Poisson manifold with vanishing Poisson structure at the point.

$$(M^n, \Pi, p) \approx (N^{2k}, \omega, p_1) \times (M_0^{n-2k}, \Pi_0, p_2)$$



The symplectic foliation on the manifold is locally a product of the induced symplectic foliation on M_0 with the symplectic leaf through x.

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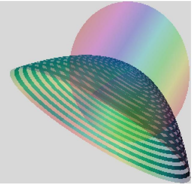

The symplectic foliation on the manifold is locally a product of the induced symplectic foliation on M_0 with the symplectic leaf through x.

The local Poisson case. Splitting Theorem.

The local structure for Poisson manifolds is given by the following:

Theorem (Weinstein)

Let (M^n, Π) *be a smooth Poisson manifold and let* p *be a point of* M *of rank* $2k$*, then there is a smooth local coordinate system* $(x_1, y_1, \ldots, x_{2k}, y_{2k}, z_1, \ldots, z_{n-2k})$ *near* p*, in which the Poisson structure* Π *can be written as*

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{ij} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j},$$

where f_{ij} *vanish at the origin.*

What is an integrable system on a Poisson manifold?

Definition

Let (M, Π) be a Poisson manifold of (maximal) rank $2r$ and of dimension n. An s-tuplet of functions $\mathbf{F} = (\mathbf{f_1}, \dots, \mathbf{f_s})$ on M is said to define a *Liouville integrable system* *on (M, Π) if*

1. $f_1, \dots, f_s$ *are independent*
2. $f_1, \dots, f_s$ *are pairwise in involution*
3. $r + s = n$

Viewed as a map, $\mathbf{F} : \mathbf{M} \to \mathbf{R^s}$ is called the *momentum map* of $(M, \Pi, \mathbf{F})$.

Some examples

- Gelfand-Ceitlin systems.
- Examples obtained from projective dynamics (we add singularities to the symplectic structure when we projectivize).
- (non-commutative) Integrable systems obtained on M/G by G-invariant functions (like geodesic flow on homogeneous spaces,Bolsinov, Jovanovic).
- Examples modelled on $T^*_b(M)$ (relation to control theory?).

Simultaneous normal Forms for integrable systems, Poisson structures and group actions

There are several problems analogous to the symplectic case that we could consider in the Poisson setting.

Normal form	**Symplectic**	**Poisson**
structure	Darboux Thm	Splitting thm
structure + r functions in inv.	Darboux-Carathéodory	?
structure+int system	Liouville-Mineur-Arnold	?
structure+group	Equivariant Darboux th.	?
structure+non-comm. int. system	Mishenko-Fomenko	?
structure+ int. system +group	Equivariant action-angle	?

We can also consider these problems for “easy” Poisson manifolds (but not as easy as symplectic). These will be the b-Poisson case.

Goals of this talk

- Action-angle coordinates (do they exist semilocally?)
- Concentrate on the particular case of b-Poisson manifolds.
- Study connection with toric actions.
- Delzant polytopes for b-cases.

A Darboux-Carathéodory theorem in the Poisson context

Theorem (Laurent, Miranda, Vanhaecke)

Let m be a point of a Poisson manifold (M, Π) of dimension n. Let $p_1, \dots, p_r$ be r functions in involution, defined on a neighborhood of m, which vanish at m and whose Hamiltonian vector fields are linearly independent at m. There exist, on a neighborhood U of m, functions $q_1, \dots, q_r, z_1, \dots, z_{n-2r}$, such that

1. *The n functions $(p_1, q_1, \dots, p_r, q_r, z_1, \dots, z_{n-2r})$ form a system of coordinates on U, centered at m;*
2. *The Poisson structure Π is given on U by*

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \sum_{i,j=1}^{n-2r} g_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \tag{2.1}$$

where each function $g_{ij}(z)$ is a smooth function on U and is independent of $p_1, \dots, p_r, q_1, \dots, q_r$.

Splitted integrable systems

Not every integrable system on a Poisson manifold can be splitted (in a compatible way with Weinstein's theorem).

Example

On $\mathbf{R}^4$, with coordinates f_1, f_2, g_1, g_2, consider:

$$\Pi = \frac{\partial}{\partial g_1} \wedge \frac{\partial}{\partial f_1} + \chi(g_2)\frac{\partial}{\partial g_2} \wedge \frac{\partial}{\partial f_2} + \psi(g_2)\frac{\partial}{\partial g_1} \wedge \frac{\partial}{\partial f_2}, \tag{2.2}$$

with $\chi(g_2)$ and $\psi(g_2)$ vanishing for $g_2 = 0$, (the rank of Π at 0 is 2).

- $\{f_1, f_2\} = 0.$
- The system would be splitted if there existed coordinates f_1, f_2, g_1, g_2 p_1, q_1, z_1, z_2, with p_1, z_1 depending only on f_1 and f_2, such that

$$\Pi = \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_1} + \phi(z_1, z_2)\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}. \tag{2.3}$$

Splitted integrable systems

The system is splitted iff the following equation has a solution

$$\chi(g_2)r'(g_2) = -\psi(g_2)c_1. \tag{2.4}$$

But this equation does not admit a smooth solution, unless $\psi(g_2)/\chi(g_2)$ is smooth at 0. Take for instance $\psi(g_2) = g_2$ and $\chi(g_2) = g_2^2$. and the system is not splitted

Laurent-Miranda

In general we can formulate the condition of an integrable system to be splitted via the Vorobjev data $(\Pi_{Vert}, \Gamma, \mathbb{F})$ associated to the Poisson structure. These data are determined in terms of the Poisson fibration over a symplectic leaf.

An action-angle theorem for Poisson manifolds

Case of regular orbits

We assume that:

1. The mapping $\mathbf{F} = (f_1, \ldots, f_s)$ defines an integrable system on the Poisson manifold (M, Π) of dimension n and (maximal) rank $2r$.
2. Suppose that $m \in M$ is a point such that it is regular for the integrable system and the Poisson structure.
3. Assume further than the integral manifold $\mathcal{F}_m$ of the foliation $X_{f_1}, \ldots X_{f_s}$ through m is compact (Liouville torus).

An action-angle theorem for Poisson manifolds

Theorem (Laurent, Miranda, Vanhaecke)

Then there exists $\mathbf{R}$-valued smooth functions $(\sigma_1, \ldots, \sigma_s)$ and $\mathbf{R}/\mathbf{Z}$-valued smooth functions $(\theta_1, \ldots, \theta_r)$, defined in a neighborhood U of $\mathcal{F}_m$ such that

1. *The functions $(\theta_1, \ldots, \theta_r, \sigma_1, \ldots, \sigma_s)$ define a diffeomorphism $U \simeq \mathbf{T}^r \times B^s$;*
2. *The Poisson structure can be written in terms of these coordinates as*

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial \sigma_i},$$

   in particular the functions $\sigma_{r+1}, \ldots, \sigma_s$ are Casimirs of Π (restricted to U);
3. *The leaves of the surjective submersion $\mathbf{F} = (f_1, \ldots, f_s)$ are given by the projection onto the second component $\mathbf{T}^r \times B^s$, in particular, the functions $\sigma_1, \ldots, \sigma_s$ depend only on the functions $f_1, \ldots, f_s$.*

The Poisson proof

1. Topology of the foliation. The fibration in a neighbourhood of a compact connected fiber is a trivial fibration by compact fibers.
2. These compact fibers are tori: We recover a $\mathbb{T}^n$-action tangent to the leaves of the foliation This implies a process of uniformization of periods.

$$\begin{aligned} \Phi \quad : \quad \mathbf{R}^r \times (\mathbf{T}^r \times B^s) &\rightarrow \mathbf{T}^r \times B^s \\ ((t_1, \ldots, t_r), m) &\mapsto \Phi^{(1)}_{t_1} \circ \cdots \circ \Phi^{(r)}_{t_r}(m). \end{aligned} \tag{2.5}$$

3. We prove that this action is Poisson (we use the fact that if Y is a complete vector field of period 1 and P is a bivector field for which $\mathcal{L}^2_Y P = 0$, then $\mathcal{L}_Y P = 0$).
4. Finally we use the Poisson Cohomology of the manifold and averaging with respect to this action to check that the action is Hamiltonian.
5. To construct action-angle coordinates we use Darboux-Carathéodory and the constructed Hamiltonian action of $\mathbb{T}^n$ to drag normal forms from a neighbourhood of a point to a neighbourhood of a fiber.

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What is an non-commutative integrable system on a Poisson manifold?

Definition

Let (M,Π) *be a Poisson manifold of dimension* n. *An* s*-uplet of functions* $\mathbf{F}=(f_1,\dots,f_s)$ *is said to be a non-commutative integrable system of rank* r *on* (M,Π) *if*

(1) $f_1,\dots,f_s$ *are independent;*

(2) *The functions* $f_1,\dots,f_r$ *are in involution with the functions* $f_1,\dots,f_s$*;*

(3) $r+s=n$*;*

(4) *The Hamiltonian vector fields of the functions* $f_1,\dots,f_r$ *are linearly independent at some point of* M*.*

Notice that $2r\leq$ *Rk* Π, *as a consequence of (4).*

Remark: The mapping $\mathbf{F}=(f_1,\dots,f_s)$ is a Poisson map on $\mathbb{R}^s$ with $\mathbb{R}^s$ endowed with a non-vanishing Poisson structure.

An action-angle theorem for non-commutative systems

Theorem (Laurent, Miranda, Vanhaecke)

Let (M, Π) be a Poisson manifold of dimension n, equipped with a non-commutative integrable system of rank r, and suppose that $\mathcal{F}_m$ is a regular Liouville torus. Then there exist $\mathbf{R}$-valued smooth functions $(p_1, \ldots, p_r, z_1, \ldots, z_{s-r})$ and $\mathbf{R}/\mathbf{Z}$-valued smooth functions $(\theta_1, \ldots, \theta_r)$, defined in a neighborhood U of $\mathcal{F}_m$, and functions such that

1. *The functions $(\theta_1, \ldots, \theta_r, p_1, \ldots, p_r, z_1, \ldots, z_{s-r})$ define a diffeomorphism $U \simeq \mathbf{T}^r \times B^s$;*
2. *The Poisson structure can be written in terms of these coordinates as*

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial p_i} + \sum_{k,l=1}^{s-r} \phi_{k,l}(z) \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_l};$$

3. *The leaves of the surjective submersion $\mathbf{F} = (f_1, \ldots, f_s)$ are given by the projection onto the second component $\mathbf{T}^r \times B^s$, in particular, the functions $p_1, \ldots, p_r, z_1, \ldots, z_{s-r}$ depend on the functions $f_1, \ldots, f_s$*

The case of b-Poisson manifolds

Definition of Poisson b-manifolds

Definition

Let (M^{2n}, Π) be an oriented Poisson manifold such that the map

$$p \in M \mapsto (\Pi(p))^n \in \Lambda^{2n}(TM)$$

is transverse to the zero section, then $Z = \{p \in M | (\Pi(p))^n = 0\}$ is a hypersurface and we say that Π is a **Poisson b-structure** on (M, Z).

This transversality condition gives stability properties for this Poisson structure.

Symplectic foliation of a Poisson b-manifold

If we use Weinstein's splitting theorem, we deduce that the symplectic foliation of a Poisson b-manifold has dense symplectic leaves of maximal dimension and codimension 2 symplectic leaves whose union is Z.

Motivation: b-objects in dimension 2

We consider pairs (M, Z) where M is a compact oriented surface and Z a union of embedded smooth curves:

Motivation: b-structures in dimension 2

Given an oriented surface S (compact or not) with a distinguished union of curves Z, we want to modify the volume form on S by making it "explode" when we get close to Z. We want this "blow up" process to be controlled.

Motivation: b-structures in dimension 2

What does "controlled" mean here?

Consider the Lie algebra $\mathfrak{g}$ to be the Lie algebra of the affine group in dimension 2.

It is a model for noncommutative Lie algebras in dimension 2 and in a basis e_1, e_2 the brackets are

$$[e_1, e_2] = e_2$$

We can naturally write this Lie algebra structure (bilinear) as the Poisson structure

$$\Pi = y\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$$

This Poisson structure is dual to the 2-form

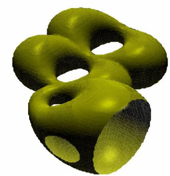
$$\omega = \frac{1}{y} dx \wedge dy$$

In this example Z is the x-axis:

In this example Z is formed by symplectic leaves of dimension 0 (points on the line). The upper and lower half-planes are symplectic leaves of dimension 2.

b-tangent bundles

A vector field in a point of the boundary has to be tangent to the boundary.



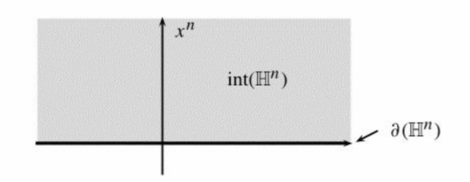


Fig. 21.1. Upper half-space.

Observe that for any point $p \in \partial(H^n_+)$, the tangent bundle at p is generated by $T_p(H^n_+) = \langle y_1 \frac{\partial}{\partial y_1}{}_p, \frac{\partial}{\partial y_2}{}_p \cdots \frac{\partial}{\partial y_n}{}_p \rangle$.

Melrose proved that there exists a vector bundle (the b-tangent bundle, ${}^bT(M)$) with sections the set of vector fields tangent to Z.

Melrose

With this idea Melrose constructed the b-cotangent bundle of this surface. This was the starting point of b-calculus for differential calculus on manifolds with boundary.

Darboux theorem

Consider the Poisson structure

$$\Pi = y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$$

It can also be interpreted as a section of $\Lambda^2({}^bT(M))$.

We also have a "Liouville" one-form interpretation for this Poisson structure.

Darboux theorem for our manifolds

We can prove a b-Darboux theorem which tells us that locally all Poisson b-manifolds look like this model. The proof uses a Moser theorem for *the complex of b-forms*.

The b-category

Thus, the moral is that b-Poisson manifolds lie between the symplectic and Poisson world and we can get some interesting results that we do not get for general Poisson manifolds.

Higher dimensions: Some compact examples.

- Let (R, π_R) be a Radko compact surface and let (S, π) be a compact symplectic surface, then $(R \times S_1, \pi_R + \pi)$ is a b-Poisson manifold of dimension 4.
- Other product structures to get higher dimensions.
- We can perturb this product structure to obtain a non-product one. For instance, S^2 with critical surface Z. Consider the Poisson structure $\Pi_1 = h\frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta}$ and the two torus $\mathbb{T}^2$ with Poisson structure $\Pi_2 = \frac{\partial}{\partial \theta_1} \wedge \frac{\partial}{\partial \theta_2}$. Consider,

$$\hat{\Pi} = h\frac{\partial}{\partial h} \wedge (\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \theta_1}) + \Pi_2.$$

Then $(S^2 \times \mathbb{T}^2, \hat{\Pi})$ is a b-Poisson manifold.

Moser's ideas

Via a path theorem, we can control perturbations that produce equivalent Poisson structures.

More examples.

- Take (N, π) be a regular Poisson manifold with dimension $2n+1$ and rank $2n$ and let X be a Poisson vector field. Now consider the product $S^1 \times N$ with the bivector field

$$\Pi = f(\theta)\frac{\partial}{\partial\theta} \wedge X + \pi.$$

This is a b-Poisson manifold as long as,

1. the function f vanishes linearly.
2. The vector field X is transverse to the symplectic leaves of N.

We then have as many copies of N as zeroes of f.

Global constructions

Induced Poisson structures

Given a b-Poisson structure Π on M^{2n} with an integrable system on it we get an induced Poisson structure on Z (critical set) which is a regular Poisson structure with an induced integrable system with symplectic leaves of codimension 1 endowed with a collection of integrable systems.

We can look for converse results.

Given a Poisson manifold Z with codimension 1 symplectic foliation $\mathcal{L}$, we want to answer the following questions:

1. Does $(Z, \Pi_{\mathcal{L}})$ extend to a b-Poisson structure on a neighbourhood of Z in M ?
2. If so to what extent is this structure unique?
3. Global results à la Radko?
4. Can we add integrable systems on it and also find a extension theorem for the integrable system?

Global constructions for higher dimensions: Going backwards...

Semilocal answer

We will thicken our regular Poisson manifold Z and we will consider a tubular neighbourhood construction:

$$\omega = p^*(\alpha_Z) \wedge \frac{df}{f} + p^*(\omega_Z)$$

Using α_Z a defining one-form for the symplectic foliation on Z and ω_Z a two form that restricts to the symplectic form on every symplectic leaf.
These forms need to satisfy more constraints in order to work.
So the answer is: Not always.
Once this construction is done, the construction of the integrable system is automatic (we just add the *defining function* for the b-manifold).

The $\mathcal{L}$-De Rham complex

Choose $\alpha \in \Omega^1(Z)$ and $\omega \in \Omega^2(Z)$ such that for all $L \in \mathcal{L}$ (symplectic foliation) such that for all $L \in \mathcal{L}$, $i_L^*\alpha = 0$ and $i_L^*\omega = \omega_L$.

Notice that

$$d\alpha = \alpha \wedge \beta, \beta \in \Omega^1(Z) \tag{3.1}$$

Therefore we can consider the complex

$$\Omega_{\mathcal{L}}^k = \Omega^K / \alpha\Omega^{k-1}$$

Consider $\Omega_0 = \alpha \wedge \Omega$ we get a short exact sequence of complexes

$$0 \longrightarrow \Omega_0 \xrightarrow{i} \Omega \xrightarrow{j} \Omega_{\mathcal{L}} \longrightarrow 0$$

By differentiation of 3.1 we get $0 = d(d\alpha) = d\beta \wedge \alpha - \beta \wedge \beta \wedge \alpha = d\beta \wedge \alpha$, so $d\beta$ is in Ω_0, i.e., $d(j\beta) = 0$.

First obstruction class

We define the **obstruction class** $c_1(\Pi_{\mathcal{L}}) \in H^1(\Omega_{\mathcal{L}})$ to be $c_1(\Pi_{\mathcal{L}}) = [j\beta]$

Notice that $c_1(\Pi_{\mathcal{L}}) = 0$ iff we can find a closed one form for the foliation.

The $\mathcal{L}$-De Rham complex

Assume now $c_1(\Pi_{\mathcal{L}}) = 0$ then, we obtain $d\omega = \alpha \wedge \beta_2$.

Second obstruction class

We define the **obstruction class** $c_2(\Pi_{\mathcal{L}}) \in H^2(\Omega_{\mathcal{L}})$ to be $c_2(\Pi_{\mathcal{L}}) = [j\beta_2]$

Main property

$c_2(\Pi_{\mathcal{L}}) = 0 \Leftrightarrow$ there exists a **closed** 2-form, ω, such that $i_L^*(\omega) = \omega_L$.

The role of these invariants

$c_1(\Pi_{\mathcal{L}}) = c_2(\Pi_{\mathcal{L}}) = 0 \Leftrightarrow$ there exists a Poisson vector field v transversal to L.

Relation of v, ω and α:

1. $i_v\alpha = 1$.
2. $i_v\omega = 0$.

The fibration is a symplectic fibration and v defines an Ehresmann connection.

Summing up,

The foliation induced by a b-Poisson structure on its critical hypersurface satisfies,

- we can choose the defining one-form α to be closed
- symplectic structure on leaves which extends to a closed 2-form ω on M

Given a symplectic foliation on a corank 1 regular Poisson manifold α and ω exists if and only if the invariants $c_1(\Pi_{\mathcal{L}})$ and $c_2(\Pi_{\mathcal{L}})$ vanish.

Question 1

How does the critical surface looks like? What about its foliation?

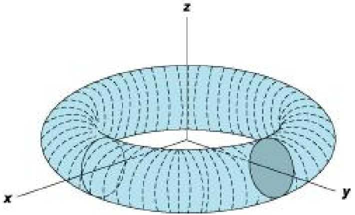
Question 2

Is every codimension one regular Poisson manifold with vanishing invariants the critical hypersurface of a b-Poisson manifold?

A symplectic variation of Tischler theorem

Theorem (Guillemin-Miranda-Pires)

If $c_1(\Pi_{\mathcal{L}})$ and $c_2(\Pi_{\mathcal{L}})$ vanish and $\mathcal{L}$ contains a compact leaf L, then M is the mapping torus of the symplectomorphism $\phi : L \to L$ determined by the flow of the Poisson vector field v.



The extension property

Theorem (Guillemin-Miranda-Pires)

Let (M^{2n+1}, Π_0) be a compact corank-1 regular Poisson manifold with vanishing invariants then there exists an extension of (M^{2n+1}, Π) to a b-Poisson manifold (U, Π). The extension is unique , up to isomorphism, among the extensions such that $[v]$ is the image of the modular class under the map:

$$H^1_{Poisson}(U) \longrightarrow H^1_{Poisson}(M^{2n+1})$$

Example

$M = \mathbb{T}^4$ and $Z = \mathbb{T}^3 \times \{0\}$. Consider on Z the codimension 1 foliation given by $\theta_3 = a\theta_1 + b\theta_2 + k$, with rationally independent $a, b, 1 \in \mathbb{R}$. Then take

$$\alpha = \frac{a}{a^2+b^2+1}\, d\theta_1 + \frac{b}{a^2+b^2+1}\, d\theta_2 - \frac{1}{a^2+b^2+1}\, d\theta_3,$$
$$\omega = d\theta_1 \wedge d\theta_2 + b\, d\theta_1 \wedge d\theta_3 - a\, d\theta_2 \wedge d\theta_3,$$

This structure can be extended to a neighbourhood of Z in M. Indeed it can be extended to the whole $\mathbb{T}^4$ by considering

$$\Pi = f(\theta_4)\frac{\partial}{\partial\theta_4} \wedge X + \pi_\omega.$$

Integrable systems on b-Poisson manifolds

In the case of Poisson b-manifolds, an integrable system is always split in a neighbourhood of Z. We then have,

Theorem (Guillemin-Miranda-Pires)

An integrable system on a Poisson b-manifold of dimension $2n$ is equivalent to an integrable system with functions $(f_1, \ldots, f_n)$ where in a neighbourhood of the critical set Z:

1. *The function f_1 can be chosen to be a defining function for Z.*
2. *The remaining first integrals $(f_2, \ldots, f_n)$ are functions on Z defining an integrable system on Z with respect to the restricted Poisson structure Π_Z.*

Integrable systems on b-Poisson manifolds

Therefore most of the classical normal form and action-angle results for symplectic manifolds also hold for Poisson b-manifolds in a neighbourhood of points in Z. In particular,

Theorem (Guillemin-Miranda-Pires)

Given an integrable system with non-degenerate singularities on a Poisson b-manifold (M, Z), there exists Eliasson-type normal forms in a neighbourhood of points in Z and the minimal rank for these singularities is 1 along Z.

Extension theorem for integrable system

Theorem

Let (M^{2n+1}, Π_0) *be a compact corank-1 regular Poisson manifold with vanishing invariants and endowed with a leafwise integrable system then there exists an extension of* (M^{2n+1}, Π) *to a b-Poisson manifold* (U, Π) *with an integrable system on it. The extension is unique , up to isomorphism, among the extensions such that* $[v]$ *is the image of the modular class under the map:*

$$H^1_{Poisson}(U) \longrightarrow H^1_{Poisson}(M^{2n+1})$$

A recipe to extend integrable systems

1. Take an integrable system on a symplectic manifold $(M, \omega, \mathbf{F})$ and consider a symplectomorphism ϕ preserving the integrable system.
2. Consider the symplectic mapping torus associated to this symplectomorphism $N^{2n+1} = \frac{M \times [0,1]}{(x,0) \sim (\phi(x),1)}$.
3. Let $f : \mathbb{S}^1 \to \mathbb{R}$ a smooth function. The bivector field $\Pi = f(\theta)\frac{\partial}{\partial \theta} \wedge X + \pi$ is a b-symplectic structure on $\mathbb{S}^1 \times N$ if the function f vanishes linearly (the manifold has as many critical components as the number of zeroes of the function f). Also the new system $\mathbf{H} = (f, \mathbf{F})$ is an integrable system on the b-symplectic manifold.

Example

Consider as b-Poisson manifold $\mathbb{R}^{2n}$ with coordinates $(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, z, t)$ $\Pi = \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + z\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial t}$. Observe that the functions $f_i = x_i \ \forall i \leq n-1$ and $f_n = z$ are pairwise in involution. Thus, these functions define an integrable system $\mathbf{F} = (x_1, \ldots, x_{n-1}, z)$.

More symmetries...

Toric manifolds and Delzant polytopes

Toric manifolds are classified by Delzant's polytopes. More specifically, the bijective correspondence between these two sets is given by the image of the moment map:

$$\begin{array}{ccc} \{\text{toric manifolds}\} & \longrightarrow & \{\text{Delzant polytopes}\} \\ (M^{2n}, \omega, \mathbb{T}^n, F) & \longrightarrow & F(M) \end{array}$$

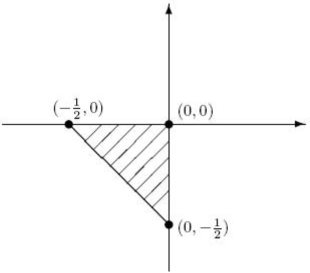


Figure : Moment map for the $\mathbb{T}^2$-action on $\mathbb{C}P^2$ given by $(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0 : z_1 : z_2] := [z_0 : e^{i\theta_1} z_1 : e^{i\theta_1} z_2]$

Equivariant Darboux-Carathéodory

Let m which is a fixed point for the action of a compact Lie group G on M. Let $p_1, \ldots, p_r$ be r G-invariant Poisson commuting functions, defined on a neighborhood of m, Hamiltonian vector fields are linearly independent at m.

Theorem (Laurent-Miranda)

There exist, in a neighborhood U of m, functions $q_1, \ldots, q_r, z_1, \ldots, z_s$, such that

1. *the n functions $(p_1, q_1, \ldots, p_r, q_r, z_1, \ldots, z_s)$ form a system of local coordinates, centered at m;*
2. *the Poisson structure Π is given in these coordinates by*

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \sum_{i,j=1}^{s} g_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \tag{4.1}$$

   where each function $g_{ij}(z)$ is a smooth function and is independent from $p_1, \ldots, p_r, q_1, \ldots, q_r$.
3. *The action is linearizable in these coordinates.*

Definition

A $\mathbb{T}^n$ action which acts on a b-symplectic manifold (M, ω) by b-symplectomorphisms is a **Hamiltonian action** if there exists a **moment map**, i.e., a map $\mu = (\mu_1, \ldots, \mu_n) : M \longrightarrow \overline{\mathbb{R}}^n$, such that

$$d\mu_i = \omega(X_i^{\#}, \cdot),$$

where $X_i^{\#}$ is the vector field generated on M by the action of the i-th circle in $\mathbb{T}^n = (\mathbb{S}^1)^n$.

In the b-toric case, we can recover information about the action from *slices* that correspond to standard Delzant polytope on the mapping torus by symplectic cutting.