ISOENERGETIC INTEGRABILITY AND CONTACT SYSTEMS WITH CONSTRAINTS

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Outline

Hamiltonian integrability

(P^{2n}, ω) - symplectic manifold.

$f \in C^\infty(P) \longmapsto X_f$, $i_{X_f}\omega = -df$ - Hamiltonian vector field.

$\{f, g\} = \omega(X_f, X_g)$ - Poisson bracket.

The Hamiltonian equations

$$\dot{x} = X_f$$

are noncommutatively completely integrable, if there are $2n - r$ almost everywhere independent integrals $f_1, f_2, \dots, f_{2n-r}$ and $f_1, \dots, f_r$ commute with all integrals

$$\{f_i, f_j\} = 0, \quad i = 1, \dots, 2n - r, \quad j = 1, \dots, r.$$

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$$\{f_i, f_j\} = 0, \quad i = 1, \dots, 2n - r, \quad j = 1, \dots, r.$$

(i) Regular compact connected invariant manifolds of the system are isotropic tori.

(ii) In a neighborhood of a regular torus, there exist canonical *generalized action-angle coordinates* $p, q, I, \varphi \mod 2\pi$

$$\omega = \sum_{i=1}^{r} dI_i \wedge d\varphi_i + \sum_{j=1}^{n-r} dp_j \wedge dq_j,$$

such that the integrals f_i, $i = 1, \ldots, r$ depend only on actions I_i and the flow is translation in angle coordinates:

$$\dot{\varphi}_1 = \omega_1(I) = \frac{\partial f}{\partial I_1}, \ldots, \dot{\varphi}_r = \omega_r(I) = \frac{\partial f}{\partial I_r}, \quad \dot{I} = \dot{p} = \dot{q} = 0.$$

- N. N. Nekhoroshev, Trans. Mosc. Math. Soc. **26** (1972)
- A. S. Mishchenko and A. T. Fomenko, Funct. Anal. Appl. **12** (1978)

Contact manifolds

$(M, \mathcal{H})$ - contact manifold.
Locally defined by a contact form: $\mathcal{H} = \ker \alpha$, $\alpha \wedge (d\alpha)^n \neq 0$.

$\mathcal{N}$ - Lie algebra of infinitesimal contact automorphisms.
Locally: $X \in \mathcal{N} \iff \mathcal{L}_X \alpha = f\alpha$.

(M, α), $\mathcal{H} = \ker \alpha$ - co-oriented (or stricly) contact manifold.

The Reeb vector field Z: $i_Z \alpha = 1, \quad i_Z d\alpha = 0$.

$$TM = \mathcal{Z} \oplus \mathcal{H}, \qquad T^*M = \mathcal{H}^0 \oplus \mathcal{Z}^0,$$

$\mathcal{Z} = \mathbb{R}Z$, $\mathcal{Z}^0$, $\mathcal{H}^0 = \mathbb{R}\alpha$ are the annihilators of $\mathcal{Z}$ and $\mathcal{H}$.
Decompositions of vector fields and 1-forms

$$X = (i_X \alpha)Z + \hat{X}, \qquad \eta = (i_Z \eta)\alpha + \hat{\eta},$$

$\hat{X}$ - horizontal, $\hat{\eta}$ - semi-basic.

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Contact Hamiltonian vector fields

$$\Phi : \mathcal{N} \longrightarrow C^\infty(M), \qquad \Phi(X) = i_X\alpha$$

infinitesimal contact automorphisms $\longleftrightarrow$ smooth functions.

$$\Phi^{-1}(f) = fZ + \alpha^\sharp(\widehat{df}).$$

$X_f = \Phi^{-1}(f)$ - contact Hamiltonian vector field, $\mathcal{L}_{X_f}\alpha = df(Z)\alpha$

$\alpha^\sharp : \mathcal{Z}^0 \to \mathcal{H}$ is the inverse of $\alpha^\flat : TM \longrightarrow \mathcal{Z}^0, \qquad X \mapsto -i_X d\alpha$

Jacobi bracket $[f, g] = \Phi[X_f, X_g] = d\alpha(X_f, X_g) + f\mathcal{L}_Z g - g\mathcal{L}_Z$

Φ is is a Lie algebra isomorphism $X_{[f,g]} = [X_f, X_g], \qquad X_{[1,f]} = [Z, X_f]$

Does not satisfy the Leibniz rule. The derivation of functions along X_f

$$\mathcal{L}_{X_f} g = [f, g] + g\mathcal{L}_Z f.$$

df is sami-basic if and only if $[1, f] = [Z, X_f] = 0$.

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Canonical coordinates

$M = \mathbb{R}^{2n+1}(x_0, \ldots, x_n, y_1, \ldots, y_n)$, $\alpha = dx_0 - \sum_{i=1}^n y_i dx_i$
The Reeb vector field: $Z = \frac{\partial}{\partial x_0}$
Contact Hamiltonian equations for a Hamiltonian f:

$$\dot{x}_0 = f - \sum_{i=1}^n y_i \frac{\partial f}{\partial x_i},$$
$$\dot{x}_i = -\frac{\partial f}{\partial y_i}, \quad \dot{y}_i = \frac{\partial f}{\partial x_i} + y_i \frac{\partial f}{\partial x_0}, \quad i = 1, \ldots, n.$$

The Jacobi bracket:

$$\begin{aligned}[f,g] \quad &= \quad \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial y_i} \right) \\ &+ \frac{\partial g}{\partial x_0}\left(f - \sum_{i=1}^n y_i \frac{\partial f}{\partial x_i} \right) - \frac{\partial f}{\partial x_0}\left(g - \sum_{i=1}^n y_i \frac{\partial g}{\partial x_i} \right)\end{aligned}$$

Non-Hamiltonian integrability

X - contact vector field (infinitesimal contact automorphisms) on $(M^{2n+1}, \mathcal{H})$. A contact equation $\dot{x} = X$ is (non-Hamiltonian) completely integrable if there is an open dense subset $M_{reg} \subset M$, a proper submersion $\pi : M^{2n+1}_{reg} \longrightarrow W^p$, and an Abelian Lie algebra $\mathcal{X}$ of symmetries such that:

(i) X is tangent to the fibers of π;

(ii) the fibers of π are orbits of $\mathcal{X}$.

The fibers of π are $(2n+1-p)$-dimensional tori with a quasi-periodic dynamics.

- V. V. Kozlov, 1996, O. I. Bogoyavlenskij, 1998, N. T. Zung, 2006

The above definition does not reflect the underlying contact structure.

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Contact integrability

We shall say that the contact equation $\dot{x} = X$ is noncommutatively contact completely integrable if if there is an open dense subset $M_{reg} \subset M$, a proper submersion $\pi : M_{reg}^{2n+1} \longrightarrow W^p$, and an Abelian Lie algebra $\mathcal{X}$ of contact symmetries such that $(M_{reg}, \mathcal{H}, \mathcal{X})$ is a complete pre-isotropic contact structure.

In the case $p = n$ we have a *regular completely integrable contact system* studied in

- A. Banyaga and P, Molino, Géométrie des formes de contact complétement intégrables de type torique, Séminare Gaston Darboux, Montpellier (1991-92), 1-25.

Complete pre-isotropic contact structures

$(M^{2n+1}, \mathcal{H})$ - contact manifold (not need to be co-oriented).

$\pi : M^{2n+1} \to W^p$ - a proper submersion, $p \geq n$.

$\mathcal{F}$ - the fibers of π.

$(M, \mathcal{H}, \mathcal{X})$ is a complete pre-isotropic contact structure if

(i) $\mathcal{F}$ is pre-isotropic, i.e., it is transversal to $\mathcal{H}$ and $\mathcal{G} = \mathcal{F} \cap \mathcal{H}$ is an isotropic subbundle of $\mathcal{H}$, or, equivalently $\mathcal{G}$ is a foliation;

(ii) $\mathcal{X}$ is an Abelian Lie algebra of infinitesimal contact automorphisms of $\mathcal{H}$, which has the fibers of π as orbits.

Theorem

Let $(M, \mathcal{H}, \mathcal{X})$ be a complete pre-isotropic contact structure related to the submersion π. Every point of M has an open, $\mathcal{X}$-invariant neighborhood U on which the contact structure can be represented by a local contact form α_U such that:

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α-Complete pre-isotropic foliations

$\mathcal{F}$ is α-complete if for any pair f_1, f_2 of first integrals of $\mathcal{F}$ (where f_i may be a constant), the bracket $[f_1, f_2]$ is also a first integral of $\mathcal{F}$ (eventually a constant).

Pseudo-orthogonal distribution: $\mathcal{E} = \mathcal{F}^{\perp}$ – locally generated by the Reeb vector field Z and the contact Hamiltonian vector fields which corresponds to the first integrals of $\mathcal{F}$.

In our case $\mathcal{E}$ is a foliation and we (locally) have a flag of foliations

$$\mathcal{G} = \mathcal{F} \cap \mathcal{H} \ \subset \ \mathcal{F} \ \subset \ \mathcal{E} = \mathcal{G}^{\perp} = \mathcal{F}^{\perp}.$$

If $\mathcal{F}$ has the maximal dimension $n+1$ then $\mathcal{F} = \mathcal{E}$ is pre-Legendrian, while $\mathcal{G}$ is a Legendrian foliation.

- P. Libermann, Differential Geometry and Its Application, **1** (1991) 57-76.
- M. Y. Pang, Trans. Amer. Math. Soc. **320** (1990), 417-455.
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Billiards within ellipsoids in $\mathbb{R}^{p,q}$

Pseudo-confocal family of quadrics

$$Q_\lambda : \quad \sum_{i=1}^{p} \frac{x_i^2}{a_i^2 + \lambda} + \sum_{i=p+i}^{p+q} \frac{x_i^2}{a_i^2 - \lambda} = 1$$

M_0^{2n-1} - contact manifold of oriented light-like lines in $\mathbb{R}^{p,q}$, $p + q = n + 1$

The set F of oriented light-like lines, tangent to the fixed $n - 1$ pseudo-confocal quadrics $Q_{\lambda_1}, \dots, Q_{\lambda_{n-1}}$ is a codimension $n - 1$ submanifold in M_0, foliated by Legendrian submanifolds of codimension 1 in F.

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Contact action-angle coordinates

Theorem (B.J. (2011))

Let F be a connected component of $\pi^{-1}(w_0)$. Then F is diffeomorphic to a $r+1$-dimensional torus $\mathbb{T}^{r+1}$, $r = 2n - p$. There exist an open $\mathcal{X}$-invariant neighborhood U of F, an $\mathcal{X}$-invariant contact contact form α on U and a diffeomorphism $\phi : U \to \mathbb{T}^{r+1} \times D$,

$$\phi(x) = (\theta, y, x) = (\theta_0, \theta_1, \dots, \theta_r, y_1, \dots, y_r, x_1, \dots, x_{2s}), \quad s = n - r,$$

where $D \subset \mathbb{R}^p$ is diffeomorphic to $W_U = \pi(U)$, such that

(i) *$\mathcal{F}|_U$ is α-complete foliation with integrals $y_1, \dots, y_r, x_1, \dots, x_{2s}$, while the integrals of the pseudo-orthogonal foliation $\mathcal{E}|_U = \mathcal{F}|_U^{\perp}$ are $y_1, \dots, y_r$.*

Theorem (the second part)

(ii) α *has the following canonical form*

$$\alpha_0 = (\phi^{-1})^*\alpha = y_0 d\theta_0 + y_1 d\theta_1 + \cdots + y_r d\theta_r + g_1 dx_1 + \cdots + g_{2s} dx_{2s},$$

where y_0 is a smooth function of y and g_i are functions of (y, x).

(iii) *the flow of X on invariant tori is quasi-periodic*

$$(\theta_0, \theta_1, \ldots, \theta_r) \longmapsto (\theta_0 + t\omega_0, \theta_1 + t\omega_2, \ldots, \theta_r + t\omega_r), \quad t \in \mathbb{R},$$

where frequencies $\omega_0, \ldots, \omega_r$ depend only on y.

We refer to local coordinates (θ, y) stated in Theorem as a generalized contact action-angle coordinates.

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Co-oriented case

$H = \ker \alpha$ - co-oriented contact structure,
Z - the Reeb vector field.

Contact Hamiltonian equations

$$\dot{x} = X_f = fZ + \alpha^\sharp(\widehat{df})$$

are noncommutatively completely integrable, if there are $2n - r$ integrals $f_1, f_2, \dots, f_{2n-r}$ (the contact Hamiltonian is either $f = f_1$ or $f = 1$), where:

$$[1, f_i] = 0, \quad [f_i, f_j] = 0, \quad i = 1, \dots, 2n - r, \quad j = 1, \dots, r.$$

and $df_1 \wedge \cdots \wedge df_{2n-r} \neq 0$ holds on an open dense set M_{reg} of M.

Theorem

Let F be a compact connected component of the level set

$$\{x \mid f_1 = c_1, \dots, f_{2n-r} = c_{2n-r}\}$$

and assume $F \subset M_{reg}$. Then

(i) *F is diffeomorphic to a $r+1$-dimensional torus $\mathbb{T}^{r+1}$. There exist a neighborhood U of F with local generalized action-angle coordinates in which α has the form $\alpha = y_0 d\theta_0 + \cdots + y_r d\theta_r + g_1 dx_1 + \cdots + g_{2s} dx_{2s}$, where y_0 is a smooth function of y and g_i are functions of (y, x).*

(ii) *The flow of X on invariant tori is quasi-periodic*

$$(\theta_0, \theta_1, \dots, \theta_r) \longmapsto (\theta_0 + t\omega_0, \theta_1 + t\omega_2, \dots, \theta_r + t\omega_r), \quad t \in \mathbb{R},$$

where frequencies $\omega_0, \dots, \omega_r$ depend only on y.

Hypersurfaces of contact type

(P, ω) - symplectic manifold, $H \in C^\infty(P)$ - Hamiltonian.

M- a regular component of an isoenergetic surface $H^{-1}(h)$.
X_H generate the characteristic line bundle $\mathcal{L}_M$ of M, the kernel of the form ω restricted to M:

$$\mathcal{L}_M = \{\xi \in T_x M \,|\, \omega(\xi, T_x M) = 0,\, x \in M\}$$

M is of contact type if there exist a 1-form α on M satisfying

$$d\alpha = j^*\omega, \quad \alpha(\xi) \neq 0,\, \xi \in \mathcal{L}_M,\, \xi \neq 0$$

where $j : M \to P$ is the inclusion. Then (M, α) is a co-oriented contact manifold with the Reeb vector field Z proportional to $X_H|_M$.

If ω is exact $\omega = d\alpha$ and $\alpha(X_H)|_M \neq 0$ then M is of contact type with respect to α.

Hypersurfaces of contact type

(P, ω) - symplectic manifold, $H \in C^\infty(P)$ - Hamiltonian.

M- a regular component of an isoenergetic surface $H^{-1}(h)$.
X_H generate the characteristic line bundle $\mathcal{L}_M$ of M, the kernel of the form ω restricted to M:

$$\mathcal{L}_M = \{\xi \in T_xM \,|\, \omega(\xi, T_xM) = 0,\ x \in M\}$$

M is of contact type if there exist a 1-form α on M satisfying

$$d\alpha = j^*\omega, \quad \alpha(\xi) \neq 0,\ \xi \in \mathcal{L}_M,\ \xi \neq 0$$

where $j : M \to P$ is the inclusion. Then (M, α) is a co-oriented contact manifold with the Reeb vector field Z proportional to $X_H|_M$.

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Isoenergetic integrability

Theorem (B.J and Vladimir Jovanovic (2012))

Let $M = H^{-1}(h)$ be a contact type hypersurface. Suppose a collection of functions $F_1 = H, \dots, F_{2n-r}$ satisfy

$$\{F_i, F_j\} = 0, \quad i = 1, \dots, 2n - r, \quad j = 1, \dots, r.$$

on M and that the restrictions $F_2|_M, \dots, F_{2n-r}|_M$ are independent. Then the Reeb flow on M is contact completely integrable in a noncommutative sense with respect to the integrals $F_2|_M, \dots, F_{2n-r}|_M$. The regular compact connected components of the invariant level sets

$$H = F_1 = h, \quad F_2 = c_2, \quad \dots, \quad F_{2n-r} = c_{2n-r} \tag{1}$$

are invariant isotropic tori of (P, ω) (or pre-isotropic tori considered on (M, α)) and the dynamics $\dot{x} = X_H$ is proportional to the quasi-periodic dynamic of the Reeb flow on M.

Partial integrability

Suppose that a Hamiltonian system $\dot{x} = X_H$ has $n-1$ commuting integrals $F_1 = H, F_2, \dots, F_{n-1}$ and an invariant relation

$$\Sigma : \qquad F_0 = 0,$$

that is, the trajectories with initial conditions on Σ stay on Σ for all time t. If Σ is of the contact type manifold and if it is invariant for all Hamiltonian flows X_{F_i}, then the compact regular components of the invariant varieties

$$F_0 = 0, \quad H = F_1 = c_1, \quad F_2 = c_2, \quad \dots, \quad F_{n-1} = c_{n-1}$$

are Lagrangian tori.

- V. Dragović, B. Gajić, Comm. Math. Phys. 256 (2006), 397-435.
- B. Jovanović, Nonlinearity 20 (2007), 221-240.
- V. Dragović, B. Gajić, B. Jovanović, IJMMP 6 (2009), 1253-1304.

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Example: Hess–Appelrot system

The phase space: $T^*SO(3)$
Local coordinates – Euler's angles (φ, θ, ψ)
Canonical symplectic form: $\omega = d(p_\varphi d\varphi + p_\theta d\theta + p_\psi d\psi)$
Hamiltonian:

$$H = \frac{1}{2}(aM_1^2 + aM_2^2 + bM_3^2 + 2cM_1M_3) + k\cos\theta,$$

where $M_1 = \frac{\sin\varphi}{\sin\theta}(p_\psi - p_\varphi\cos\theta) + p_\theta\cos\varphi$,
$M_2 = \frac{\cos\varphi}{\sin\theta}(p_\psi - p_\varphi\cos\theta) - p_\theta\sin\varphi, \qquad M_3 = p_\varphi$.

Invariant relation: $\Sigma: \qquad F_0 = M_3 = p_\varphi = 0$

Integrals: $F_1 = H, F_2 = p_\psi$

$$\{F_0, F_1\} = 0|_\Sigma, \quad \{F_0, F_2\} = 0, \quad \{F_1, F_2\} = 0$$

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$$\{F_0, F_1\} = 0|_\Sigma, \quad \{F_0, F_2\} = 0, \quad \{F_1, F_2\} = 0$$

The Hamiltonian vector field of F_0 is $X_{F_0} = \partial/\partial\varphi$

$$(p_\varphi d\varphi + p_\theta d\theta + p_\psi d\psi)(X_{F_0}) = p_\varphi \equiv 0|_\Sigma$$

Σ is of contact type with respect to α:

$$\alpha = p_\varphi d\varphi + p_\theta d\theta + p_\psi d\psi + d\varphi, \qquad \alpha(X_{F_0}) \equiv 1.$$

Therefore, compact regular level sets

$$F_0 = 0, \quad F_1 = H = h, \quad F_2 = c$$

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Contact submanifolds

A *contact submanifold* of the contact manifold $(M, \mathcal{H}_M)$ is a triple $(N, \mathcal{H}_N, j)$, where $(N, \mathcal{H}_N)$ is a contact manifold and $j : N \to M$ is an embedding such that $j_*^{-1}(\mathcal{H}_M) = \mathcal{H}_N$.

Let (M, α) be a co-oriented contact manifold and $j : N \to M$ an embedding. If we define

$$\mathcal{H}_N = \{X \in TN \,|\, j_*(X) \in \mathcal{H}_M\} = j_*^{-1}(\mathcal{H}_M),$$

then $\mathcal{H}_N = \ker(j^*\alpha)$.

$(N, j^*\alpha)$ is a contact co-oriented submanifold of (M, α), if N is transverse to $\mathcal{H}_M$ and if $dj^*\alpha$ is non-degenerate on $\ker(j^*\alpha)$.

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Dirac's construction

Let (M, α) be a $(2n+1)$-dimensional co-oriented contact manifold, $G_1, \ldots, G_{2k}$ smooth functions on M,

$$N = \{x \in M \,|\, G_1(x) = \ldots = G_{2k}(x) = 0\}, \tag{2}$$

and $j : N \to M$ be the corresponding embedding.

(a) If $[1, G_j] = 0|_N$, $j = 1, \ldots, 2k$ and

$$\det([G_j, G_i]) \neq 0|_N \tag{3}$$

then $(N, j^*\alpha)$ is a contact submanifold of (M, α) with the Reeb vector field that is the restriction of the the Reeb vector field Z of (M, α).

(b) Let f be a smooth function on M and

$$W_f = Y_f - \sum_{i=1}^{2k} \lambda_i Y_{G_i}.$$

Then the system

$$dG_j(W_f) = Y_f(G_j) - \sum_i \lambda_i Y_{G_i}(G_j) = 0 \qquad j = 1, \dots, 2k \tag{4}$$

has a unique solution $\lambda_1 = \lambda_1(f), \dots, \lambda_{2k} = \lambda_{2k}(f)$ on N. For the given multipliers,

$$Y_f^* = W_f$$

is the contact Hamiltonian vector field of the function f restricted to N. If g is any smooth function on M, the Jacobi bracket between the restrictions of f and g to N is given by

$$[f|_N, g|_N]_N = [f, g] + \sum_{i,j} [G_i, g] A_{ij} [G_j, f], \tag{5}$$

where A_{ij} is the inverse of the matrix $([G_i, G_j])$.

Contact sphere

$\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$, $z_j = x_j + iy_j$, $(j = 0, \dots, n)$.

$$S^{2n+1}: \quad F = 1, \qquad F(z, \bar{z}) = \sum_{j=0}^{n} |z_j|^2.$$

$$\alpha = \frac{i}{8} \sum_{j=0}^{n} a_j (z_j d\bar{z}_j - \bar{z}_j dz_j) = \frac{1}{4} \sum_{j=0}^{n} a_j (x_j dy_j - y_j dx_j).$$

(S^{2n+1}, α) is a co-oriented contact manifold with the Reeb vector field

$$Z = 4i \sum_{j=0}^{n} \frac{1}{a_j} \left(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right).$$

The Reeb flow induced is completely contact integrable by means of commuting integrals

$$f_j(z) = |z_j|^2, \quad Y_j = \frac{4i}{a_j} \left(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right), \quad j = 0, \dots, n$$

Reduction to Brieskorn manifolds

Let $G(z) = \sum_{j=0}^{n} z_j^{a_j}$. The set

$$B = \{z \in \mathbb{C}^{n+1} \;:\; F(z,\bar{z}) = 1,\; G(z) = 0\}$$

is known as Brieskorn manifold and (B, α) is a co-oriented contact manifold with the Reeb vector field $Z|_B$.

- R. Lutz, C. Meckert, C.R. Acad. Sci. Paris Ser. A-B **282** (1976).

From a point of view of the construction presented in the previous section, note that

$$Z(G) = 4i \cdot G,$$

implying $$[G_1, 1] = [G_2, 1] = 0|_B,$$

$G_1 = \frac{1}{2}\sum_{j=0}^{n}\left(z_j^{a_j} + \bar{z}_j^{a_j}\right) = \Re(G), \quad G_2 = \frac{1}{2i}\sum_{j=0}^{n}\left(z_j^{a_j} - \bar{z}_j^{a_j}\right) = \Im(G).$
Also

$$[G_1, G_2] = \mu = 2\sum_{j=0}^{n} a_j |z_j|^{2(a_j-1)} = 2\sum_{j=0}^{n} a_j f_j^{a_j-1} \neq 0.$$

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Theorem (B.J and Vladimir Jovanovic (2012))

Let f and g be integrals of the Reeb vector field $Z|_B$. Then

$$[f,g]_B = [f,g] + \frac{df(V_2)dg(V_1) - df(V_1)dg(V_2)}{\mu},$$

where $[\cdot,\cdot]_B$ is the Jacobi bracket on (B,α), $[\cdot,\cdot]$ is the Jacobi bracket on (S^{2n+1},α) and

$$V_1 = 2i\sum_{j=0}^{n}\left(\bar{z}_j^{a_j-1}\frac{\partial}{\partial z_j} - z_j^{a_j-1}\frac{\partial}{\partial \bar{z}_j}\right),\ V_2 = -2\sum_{j=0}^{n}\left(\bar{z}_j^{a_j-1}\frac{\partial}{\partial z_j} + z_j^{a_j-1}\frac{\partial}{\partial \bar{z}_j}\right)$$

$$[f_j, f_k]_B = \frac{8i}{\mu}\left[\bar{z}_j^{a_j} z_k^{a_k} - z_j^{a_j}\bar{z}_k^{a_k}\right] \neq 0,$$

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"Exotic" spheres

$a_0 = p, \quad a_1 = \cdots = a_n = 2$, where $n = 2m+1$ and $p \equiv \pm 1 \pmod 8$.

$B_p \approx S^{4m+1}$ E. Brieskorn, Invent. Math. **2** (1966).

For $p_1 \neq p_2$, the contact structures $\mathcal{H}_{p_1} = \ker \alpha_{p_1}$ and $\mathcal{H}_{p_1} = \ker \alpha_{p_1}$ are not isomorphic. Ustilovsky, Internat. Math. Res. Notices (1999).

The proof is based on the study of periodic trajectories of the Reeb flow of the perturbed contact form $\frac{1}{H}\alpha_p$, which is equal to the contact flow

$$\dot{z} = Y_H^* \tag{6}$$

on (B_p, α_p), where

$$H = F + \sum_{j=1}^{m} \epsilon_j g_j, \qquad 0 < \epsilon_j < 1, \quad j = 1, \dots, m,$$

$$g_j = i(\bar{z}_{2j} z_{2j+1} - z_{2j}\bar{z}_{2j+1}) = 2(y_{2j}x_{2j+1} - y_{2j+1}x_{2j}).$$

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From a point of view of integrability, we can consider the contact flow of H as an integrable perturbation of the Reeb flow.

Theorem

The flow of Y^*_H is completely integrable in a noncommutative sense. Generic invariant pre-isotropic tori are of dimension $m+1$, spanned by the Reeb flow and the contact flows of integrals $g_1, \dots, g_m$.

- B. J, Noncommutative integrability and action angle variables in contact geometry, to appear in J. Symplectic Geometry (2012), arXiv:1103.3611 [math.SG]
- B. J, Vladimir Jovanovic, Contact flows and integrable systems, (2012).

Let (M, α) be a co-oriented contact manifold with a complete pre-isotropic contact structure defined by commuting infinitesimal automorphisms $\mathcal{X}$ of α, such that the associated Reeb vector field Z is a section of $\mathcal{F} = \ker \pi_*$. We refer to a triple $(M, \alpha, \mathcal{X})$ with the above property as a complete pre-isotropic structure of the Reeb type.

The associated foliation $\mathcal{F} = \ker \pi_*$ is α-complete.

- N. N. Nekhoroshev, Action-angle variables and their generalization. Trans. Mosc. Math. Soc. **26** (1972)
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Let $(M,\alpha,\mathcal{X})$ be a complete pre-isotropic structure of the Reeb type and assume the fibers of are connected. Suppose that the intersection of torodial domains U_i and U_j is connected. Then on $U_i\cap U_j$ we have the following transition formulas:

$$\theta^j_\nu=\sum_{\mu=0}^{r} M^{ij}_{\nu\mu}(\theta^i_\mu+F^{ij}_\mu(y^i,x^i)),$$

$$y^j_\nu=\sum_{\mu=0}^{r} K^{ij}_{\nu\mu}y^i_\mu, \qquad \nu=0,\dots,r,$$

$$x^j_a=X^{ij}_a(y^i,x^i), \qquad a=1,\dots,2s,$$

where matrixes $K^{ij}=(K^{ij}_{\nu\mu})$ and $M^{ij}=(K^{ij}_{\nu\mu})$ belong to $GL(r+1,\mathbb{Z})$, $M=(K^T)^{-1}$, and functions $X^{ij}_a(y^i,x^i)$, $F^{ij}_\nu(y^i,x^i)$ satisfy

$$g^i_a=\sum_{b=1}^{2s} g^j_b\frac{\partial X^{ij}_b}{\partial x^i_a}, \qquad \sum_{b=1}^{2s} g^j_b\frac{\partial X^{ij}_b}{\partial y^i_k}+\sum_{\nu=0}^{r} y^i_\nu\frac{\partial F^{ij}_\nu}{\partial y^i_k}=0.$$

Theorem

Let $(M, \alpha, \mathcal{X})$ be a complete pre-isotropic structure of the Reeb type and let $W' \subset W$, dim W' = dim W *be a connected compact submanifold (with a smooth boundary) such that*
(i) $\pi : M' \to W'$ is a trivial principal $\mathbb{T}^{r+1}$ bundle, $M' = \pi^{-1}(W')$.
(ii) There exist everywhere independent functions $\bar{x}_1, \ldots, \bar{x}_{2s}$ defined is some neighborhood of W' satisfying:

$$\langle dx_1, \ldots, dx_{2s} \rangle \cap \mathcal{E}^0 = 0,$$

where $x_a = \bar{x}_a \circ \pi$ and $\mathcal{E} = \mathcal{F}^\perp$ is the pseudo-orthogonal foliation of $\mathcal{F}$. Then there exist global action-angle variables $(\theta_0, \ldots, \theta_r, y_0, \ldots, y_r)$ and functions $\bar{g}_1, \ldots, \bar{g}_{2s} : W' \to \mathbb{R}$ such that the contact form α on M' reads

$$\alpha_0 = y_0 d\theta_0 + \cdots + y_r d\theta_r + \pi^*(\bar{g}_1 d\bar{x}_1 + \cdots + \bar{g}_{2s} d\bar{x}_{2s}).$$