A puzzle of 6 hyperelliptic Jacobians and separation of variables for some integrable systems

Workshop: Integrability in Dynamical Systems and Control

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An example: The Frahm–Manakov top on $so(n)$

$$\dot{M} = [M, \Omega], \qquad M_{ij} = \frac{a_i - a_j}{b_i - b_j}\Omega_{ij}, \quad 1 \le i << \le n, \quad M, \Omega \in so(n)$$

A Lax representation with a spectral parameter λ (Dubrovin and Manakov):

$$\dot{L}(\lambda) = [L(\lambda), U(\lambda)], \quad L = M + \lambda A, \quad U = \Omega + \lambda B, \quad A = diag(a_1, \dots, a_n)$$

• **Theorem** The solutions M are linearized on the Jacobian variety of the spectral curve $\mathcal{S} = \{|L(\lambda) - \mu I| = 0\}$.
• For $n > 3$ the genus g of $\mathcal{S}$ is bigger than the dimension d of generic invariant tori:

n	g	d
3	1	1
4	3	2
5	6	4
6	9	6

• How to relate the Jacobian variety of $\mathcal{S}$ and the complex invariant tori ?

• F. Schottky (1891): There is a *linear* relation between the Frahm-(Manakov) top on $so(4)$ and *the Clebsch integrable case* of the Kirchoff equations on $e(3) = \{K, p\} = \mathbb{R}^6$:

$$\dot{K} = K \times \frac{\partial H}{\partial K} + p \times \frac{\partial H}{\partial p},$$
$$\dot{p} = p \times \frac{\partial H}{\partial K}$$

with the Hamiltonian

$$H_1 = \frac{1}{2}(c_1 K_1^2 + c_2 K_2^2 + c_3 K_3^2) + \frac{1}{2}(b_1 p_1^2 + b_2 p_2^2 + b_3 p_3^2),$$
$$\frac{b_1 - b_2}{c_3} + \frac{b_2 - b_3}{c_1} + \frac{b_3 - b_1}{c_2} = 0$$

and with the 3×3 Lax representation with an *elliptic* spectral parameter

$$\dot{L}(r) = [L(r), A(r)], \quad L_{ij}(r) = \varepsilon_{ijk}\left(\sqrt{r - c_k}\, K_k + \sqrt{(r - c_i)(r - c_j)}\, p_k\right).$$

The spectral curve $|L(r) - \mathbf{I}y| = 0$:

$$C : \{y^2 = h_2 r^2 + h_1 r + h_0 + 2h_3\sqrt{(r - c_1)(r - c_2)(r - c_3)}\}.$$

The spectral curve

$$C: \left\{ y^2 = h_2 r^2 + h_1 r + h_0 + 2h_3 \sqrt{\underbrace{(r-c_1)(r-c_2)(r-c_3)}_{\Psi(r)}} \right\}$$

covers the elliptic curve $E: \{w^2 = (r-c_1)(r-c_2)(r-c_3)\}$.
The 2-fold covering $C \to E$ is ramified at 4 points $Q_i = (s_i, \sqrt{\Psi(s_i)})$,
s_i being the roots of the polynomial

$$\psi(r) = (h_2 r^2 + h_1 r + h_0)^2 - 4h_3^2 \Psi(r) = (r-s_1)(r-s_2)(r-s_3)(r-s_4)\,.$$

Hence genus(C)=3 (The Riemann–Hurwitz formula)
C admits an involution $\sigma: C \to C$ with 4 fixed points $Q_1, \dots, Q_4$.

- L. Heine (1983): The complex (2-dim.) generic invariant varieties of the Clebsch system are open subsets of the Prym varieties $\mathrm{Prym}(C/\sigma) \subset \mathrm{Jac}(C)$ (the anti-symmetric part of $\mathrm{Jac}(C)$ under σ).

- On the other hand, F. Kötter (1891) integrated the Clebsch system by using the genus 2 hyperelliptic functions (no Pryms !).

Ueber die Bewegung eines festen Körpers in einer Flüssigkeit.

(Von Herrn *Fritz Kötter*.)

α denselben Werth beilegen und dann die Wurzeln $\sqrt{s_\beta - c_\alpha}$ entsprechend der eben angegebenen Bedingung auswählen. Ferner soll gesetzt werden:

$$\text{(24.)}\quad \begin{cases} \xi_\alpha = x_\alpha\left(\dfrac{\sqrt{(s_1-c_1)(s_1-c_2)(s_1-c_3)}}{\sqrt{s_1-c_\alpha}\sqrt{\psi'(s_1)}} + i\,\dfrac{\sqrt{(s_2-c_1)(s_2-c_2)(s_2-c_3)}}{\sqrt{s_2-c_\alpha}\sqrt{\psi'(s_2)}}\right) \\ \qquad\qquad + y_\alpha\left(\dfrac{\sqrt{s_1-c_\alpha}}{\sqrt{\psi'(s_1)}} + i\,\dfrac{\sqrt{s_2-c_\alpha}}{\sqrt{\psi'(s_2)}}\right), \\ \eta_\alpha = x_\alpha\left(\dfrac{\sqrt{(s_1-c_1)(s_1-c_2)(s_1-c_3)}}{\sqrt{s_1-c_\alpha}\sqrt{\psi'(s_1)}} - i\,\dfrac{\sqrt{(s_2-c_1)(s_2-c_2)(s_2-c_3)}}{\sqrt{s_2-c_\alpha}\sqrt{\psi'(s_2)}}\right) \\ \qquad\qquad + y_\alpha\left(\dfrac{\sqrt{s_1-c_\alpha}}{\sqrt{\psi'(s_1)}} - i\,\dfrac{\sqrt{s_2-c_\alpha}}{\sqrt{\psi'(s_2)}}\right), \end{cases}$$

$$\text{(25.)}\quad d_\alpha = \frac{\dfrac{\sqrt{s_3-c_\alpha}}{\sqrt{\psi'(s_3)}} + i\,\dfrac{\sqrt{s_4-c_\alpha}}{\sqrt{\psi'(s_4)}}}{\dfrac{\sqrt{s_1-c_\alpha}}{\sqrt{\psi'(s_1)}} + i\,\dfrac{\sqrt{s_2-c_\alpha}}{\sqrt{\psi'(s_2)}}}, \quad -d_\alpha^{-1} = \frac{\dfrac{\sqrt{s_3-c_\alpha}}{\sqrt{\psi'(s_3)}} - i\,\dfrac{\sqrt{s_4-c_\alpha}}{\sqrt{\psi'(s_4)}}}{\dfrac{\sqrt{s_1-c_\alpha}}{\sqrt{\psi'(s_1)}} - i\,\dfrac{\sqrt{s_2-c_\alpha}}{\sqrt{\psi'(s_2)}}}.$$

Setzt man nun

$$\text{(30.)}\quad Z_\beta = \sqrt{z_\beta(z_\beta-d_1^2)(z_\beta-d_2^2)(z_\beta-d_3^2)\left(\frac{z_\beta}{d_1^2 d_2^2 d_3^2}-1\right)} \qquad (\beta = 1, 2),$$

Let

$$\psi(r) = (\underbrace{h_2 r^2 + h_1 r + h_0}_{g(r)})^2 - 4h_3^2\Psi(r) = (r - s_1)(r - s_2)(r - s_3)(r - s_4)\,.$$

F. Kötter (1891) gave a solution of the Clebsch system in terms of theta-functions of the genus 2 curve

$$\Gamma = \{w^2 = z(z - d_1^2)(z - d_2^2)(z - d_3^2)(z - d_1^2 d_2^2 d_3^2)\}$$

$$d_\alpha = \frac{\frac{\sqrt{s_3 - c_\alpha}}{\sqrt{\psi'(s_3)}} + i\frac{\sqrt{s_4 - c_\alpha}}{\sqrt{\psi'(s_4)}}}{\frac{\sqrt{s_1 - c_\alpha}}{\sqrt{\psi'(s_1)}} + i\frac{\sqrt{s_2 - c_\alpha}}{\sqrt{\psi'(s_2)}}}, \quad \alpha = 1, 2, 3$$

- Is this curve well defined ?

Observation: Changing signs of the roots may only lead to the transformations

$$(d_1^2, d_2^2, d_3^2) \to (1/d_1^2, 1/d_2^2, 1/d_3^2) \text{ or } \to (d_1^2, d_2^2, 1/d_3^2) \quad \text{or} \to (d_1^2, 1/d_2^2, 1/d_3^2),$$

which, however, do not change the absolute invariants of Γ !

Changing the order of s_i do will change the invariants of the curve.

The 3 Kötter curves

$$\Gamma = \{ w^2 = z(z - d_1^2)(z - d_2^2)(z - d_3^2)(z - d_1^2 d_2^2 d_3^2) \}$$

where

$$d_\alpha^2 + \frac{1}{d_\alpha^2} = 2\frac{S^\alpha_{(14)(23)} + S^\alpha_{(24)(13)}}{S^{(\alpha)}_{(12)(34)}}, \quad \alpha = 1, 2, 3$$

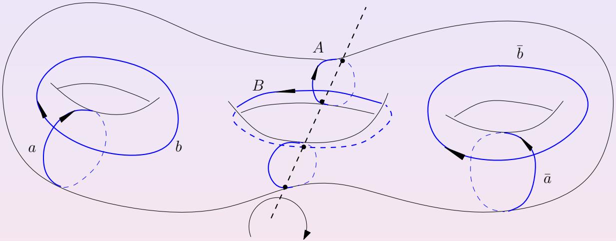
the permutations of the terms $S^\alpha_{(ij)(kl)}$ are

$$S^\alpha_{(12)(34)} = (s_1 - s_2)(s_3 - s_4)(2c_\alpha^2 - c_\alpha(s_1 + s_2 + s_3 + s_4) + s_1 s_2 + s_3 s_4 + 2\psi(c_\alpha)$$
$$S^\alpha_{(14)(23)} = (s_1 - s_4)(s_2 - s_3)(2c_\alpha^2 - c_\alpha(s_1 + s_2 + s_3 + s_4) + s_1 s_4 + s_2 s_3 + 2\psi(c_\alpha)$$
$$S^\alpha_{(24)(13)} = (s_2 - s_4)(s_1 - s_3)(2c_\alpha^2 - c_\alpha(s_1 + s_2 + s_3 + s_4) + s_1 s_3 + s_2 s_4 + 2\psi(c_\alpha)$$

So, there are 3 birationally non-equivalent genus 2 Kötter curves $Gamma_1, \Gamma_2, \Gamma_3$.
How $\mathrm{Prym}(C/\sigma) \subset \mathrm{Jac}(C)$ *and* $\mathrm{Jac}(\Gamma_\alpha)$ *are related ?*

Extracting the Prym variety from Jac(C) (following J. Fay)

The involution $\sigma: C \to C$ with 4 fixed points $Q_1, \ldots, Q_4$.
Then C is a 2-fold covering $\pi: C \to E$, $E = C/\sigma$ (elliptic curve), ramified at $Q_1, \ldots, Q_4$.



The involution σ rotates the whole surface by π and has 4 fixed points $Q_1, \ldots, Q_4$

Action on cicles: $\sigma(a) = -\bar{a}, \quad \sigma(A) = -A, \quad \sigma(b) = -\bar{b}, \quad \sigma(B) = -B,$

Let $u, w, \bar{u}$ be the corresponding *normalized* holomorphic differentials, such that

$$\begin{bmatrix} \oint_a u & \oint_A u & \oint_{\bar{a}} u \\ \oint_a w & \oint_A w & \oint_{\bar{a}} w \\ \oint_a \bar{u} & \oint_A \bar{u} & \oint_{\bar{a}} \bar{u} \end{bmatrix} = \mathbf{I}, \quad R = \begin{bmatrix} \oint_b u & \oint_B u & \oint_{\bar{b}} u \\ \oint_b w & \oint_B w & \oint_{\bar{b}} w \\ \oint_b \bar{u} & \oint_B \bar{u} & \oint_{\bar{b}} \bar{u} \end{bmatrix}.$$

Extracting the Prym variety from Jac(C) (II)

Apply 2 transformations of degree 2 on the period matrix:

$$(\mathbf{I}\, R) = (V_1 \cdots V_6) \longrightarrow \left\{ \begin{array}{l} V_4' = V_4 + V_6 \\ V_6' = V_4 - V_6 \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} V_1' = V_1 + V_3 \\ V_3' = V_1 - V_3 \end{array} \right\}$$

$$\longrightarrow \begin{bmatrix} 2 & 0 & 0 & 2\pi & 2p & 0 \\ 0 & 1 & 0 & 2p & 2P & 0 \\ 0 & 0 & 1 & 0 & 0 & \tau \end{bmatrix} \leftarrow \text{complete splitting}$$

Hence Jac(C) contains the (2,1) polarized Abelian subvariety Prym(C/σ) with the normalized period matrix

$$\Lambda = \begin{bmatrix} 2 & 0 & \Pi \\ 0 & 1 & \end{bmatrix}, \quad \Pi = \begin{pmatrix} 2\pi & 2p \\ 2p & 2P \end{pmatrix}$$

Relations between Prym(C/σ) and different 2-dim. Jacobians

The simplest transformations

$$\begin{bmatrix} 1 & 0 & \frac{1}{2}\Pi \\ 0 & 1 & \frac{1}{2}\Pi \end{bmatrix} \cong \begin{bmatrix} 2 & 0 & \Pi \\ 0 & 2 & \Pi \end{bmatrix} \to^{2:1} \Lambda = \begin{bmatrix} 1 & 0 & \Pi \\ 0 & 2 & \Pi \end{bmatrix} \to^{2:1} \begin{bmatrix} 1 & 0 & \Pi \\ 0 & 1 & \Pi \end{bmatrix}$$

The general case:

$$\Lambda \to \Lambda\, S\, D_1\, T,$$

$$S \in \mathsf{Sp}\,(4, \mathbb{Z}, (1,2))\,, \quad T \in \mathsf{Sp}\,(4, \mathbb{Z})\,, \quad D_1 = \mathrm{diag}(2,1,1,1),$$

$$S J_{(1,2)} S^T = J_{(1,2)}, \quad J_{(1,2)} = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

Def. $A, B \in Sp(4, \mathbb{Z}, (1,2))$ are D_1-*equivalent*, when the principally polarized period matrices $\Lambda A D_1$, $\Lambda B D_1$ are symplectically equivalent.

Proposition

There are at least 3 equivalence clases represented by matrices

$$S_1 = \begin{bmatrix} 0 & 0 & -1 & -2 \\ 1 & -1 & -1 & -2 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & 0 & -1 & -2 \\ 1 & -1 & -1 & -2 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad S_3 = \mathrm{Id},$$

which give 3 non-equivalent *principally polarized* Abeian varieties $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ (i.e., Jacobians) over $\mathrm{Prym}(C, \sigma)$ with the period matrices $\Lambda\, S_i\, D_1\, T$

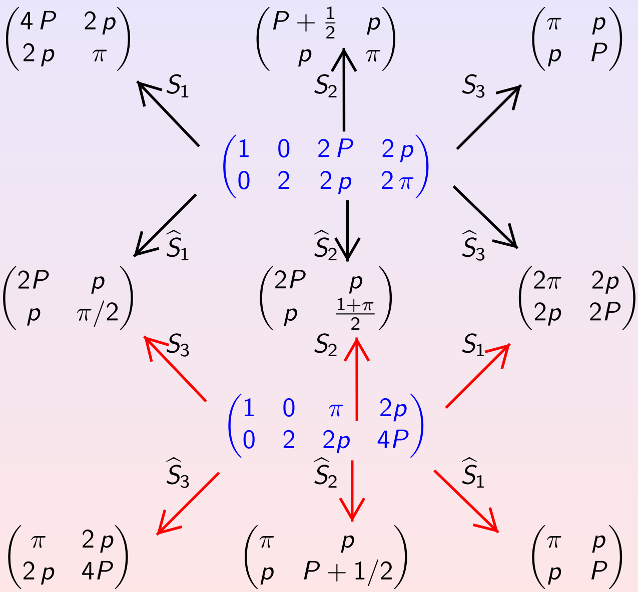
$$\widetilde{\Omega}_1 \sim \begin{bmatrix} 1 & 0 & 4\,P & 2\,p \\ 0 & 1 & 2\,p & \pi \end{bmatrix}, \quad \widetilde{\Omega}_2 \sim \begin{bmatrix} 1 & 0 & P + \frac{1}{2} & p \\ 0 & 1 & p & \pi \end{bmatrix},$$

$$\widetilde{\Omega}_3 \sim \begin{bmatrix} 1 & 0 & 2\,P & 2\,p \\ 0 & 1 & 2\,p & 2\,\pi \end{bmatrix}$$

• The equivalence classes $\{S_1\}, \{S_2\}, \{S_3\}$ form a subgroup of $\mathrm{Sp}(4, \mathbb{Z}, (1,2))$.
• Another 3 ("lowering") transformations $\Lambda \to \Lambda\, \widehat{S}_i\, D_2^{-1}\, T$, $\widehat{S}_i \in Sp\,(4, \mathbb{Z}, (1,2))$ give 3 non-equivalent Abelian varieties "below" $\mathrm{Prym}(C/\sigma)$.

Main Theorem $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\} = \{\mathrm{Jac}(\Gamma_1), \mathrm{Jac}(\Gamma_2), \mathrm{Jac}(\Gamma_3)\}$.

The puzzle of six Jacobians (II)



A numerical test (I)

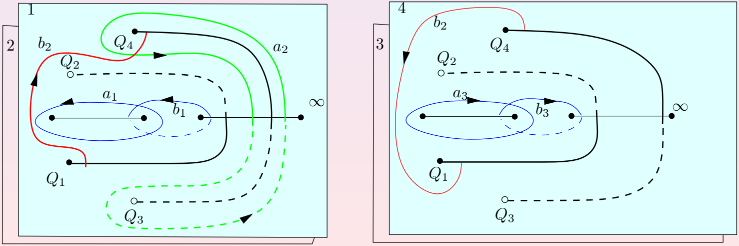
The 2-fold covering: genus 3 curve $\to$ elliptic curve (ramified at $Q_1, \ldots, Q_4$)

$$C: \{y^2 = x^2 + 3x + 5 + \sqrt{4x^3 - 4x}\} \to E: \{z^2 = 4x(x^2 - 1)\}$$

and the holomorphic differentials on C:

$$w_1 = \frac{dx}{y(y^2 - x^2 - 3x - 5)}, \quad w_2 = \frac{x\,dx}{y(y^2 - x^2 - 3x - 5)}, \quad \omega = \frac{dx}{y^2 - x^2 - 3x - 5},$$

The curve C as 2-fold cover of E with anti-symmetic cycles on it:
$\sigma(a_1) = -a_3,\ \sigma(a_3) = -a_1,\ \sigma(a_2) = -a_2$



black lines = cuts   colour lines = cycles

solid/dashed lines on the upper/lower sheets

A numerical test(II)

To calculate the periods of w_1, w_2, introduce the elliptic parametrization

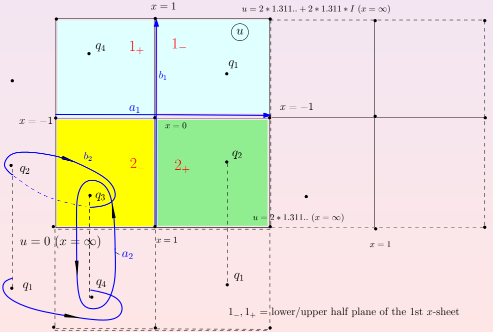
$$x = \wp(u, g_2 = 4, g_3 = 0), \quad \sqrt{4x^3 - 4x} = \wp'(u, g_2 = 4, g_3 = 0), \quad u \in \mathbb{C}$$

Then, on C, $y^2 = \wp(u)^2 + 3\wp(u) + 5 + \wp'(u)$, $dx = \wp'(u)\,du$, and

$$w_1 = \frac{du}{\sqrt{\wp(u)^2 + 3\wp(u) + 5 + \wp'(u)}}, \quad w_2 = \frac{\wp(u)\,du}{\sqrt{\wp(u)^2 + 3\wp(u) + 5 + \wp'(u)}}.$$

The elliptic curve E (its paralelogram of periods) and the branch points q_i

cycles are shown with blue curves



A numerical test(III)

With the above choice of cycles, the period matrix of Jac(C) is[1]:

$$\mathcal{A} = \begin{pmatrix} 0.3409731370 & 0.05372000635\, I & 0.3409731370 \\ -0.1676806592 & 0.6218457230\, I & -0.1676806592 \\ 0.655514404 & 0 & -0.655514404 \end{pmatrix},$$

$$\mathcal{B} = \begin{pmatrix} 0.2557651545\, I & 0.2047172550 & 0.2557651545\, I \\ 0.1075742777\, I & -0.5928537135 & 0.1075742777\, I \\ 0.6555144040\, I & 0 & -0.6555144040\, I \end{pmatrix}$$

$$\mathcal{A}^{-1}\mathcal{B} = R = \begin{pmatrix} 0.846695699\, I & 0.36000339 & -0.1533043007\, I \\ 0.3599648533 & 0.759227705\, I & 0.359964853 \\ -0.15330430078\, I & 0.36000339249 & 0.8466956992\, I \end{pmatrix}.$$

Hence, the period matrix of Prym(C/σ) is

$$\Lambda = \begin{pmatrix} 2 & 0 & 1.386782796\, I & 0.7200067850 \\ 0 & 1 & 0.7199297066 & 0.75922770504\, I \end{pmatrix}.$$

[1] Thanks to H. Braden for confirming this !

The Prym period matrix:

$$\Lambda = \begin{pmatrix} 2 & 0 & 1.386782796 * I & 0.7200067850 \\ 0 & 1 & 0.7199297066 & 0.75922770504 * I \end{pmatrix}.$$

The period matrix of the Jacobian of the Kötter hypereliptic curve:

$$\begin{pmatrix} 1 & 0 & 1 + 2.28721934 * I & -1 - 1.36067850 * I \\ 0 & 1 & -1 - 1.3606786 * I & 1.4474683 * I \end{pmatrix}.$$

Observation:
The transformation from $\mathrm{Prym}(C/\sigma)$ *to* $\mathrm{Jac}(\Gamma)$ *is given* by $\Lambda \to \Lambda S_1 D_1$,

$$S_1 D_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \det S_1 D_1 = 2.$$

That is, in this example, the Jacobian of the Kötter curve Γ *is a 2-fold (unramified) covering of* $\mathrm{Prym}(C/\sigma)$.
How to check this? Compare the 3 absolute j-invariants of $\Lambda S_1 D_1$ (analytic) and of $\mathrm{Jac}(\Gamma)$ (algebraic).

$$j_1 = 2.04896, \quad j_2 = -2.57195, \quad j_3 = 0.$$

Absolute invariants in terms of theta-constants

$$\boxed{j_1 = 144\frac{J_4}{J_2^2}, \quad j_2 = -1728\frac{(J_4 J_2 - 3J_6)}{J_2^3}, \quad j_3 = 486\frac{J_{10}}{J_2^5},}$$

$$J_2 = 48\pi^{12}\,\frac{\displaystyle\sum_{15\text{ terms}}\ \prod_{k=1,\dots,6,\ \sum[\varepsilon_k]=0}\theta^4[\varepsilon_k]}{\left(\displaystyle\prod_{6\text{ odd }[\delta]}\theta_1[\delta]\right)^2}$$

$$J_4 = 72\pi^{24}\,\frac{\displaystyle\sum_{10\text{ even }[\varepsilon]}\theta^8[\varepsilon_k]\prod_{10\text{ even }[\varepsilon]}\theta^4[\varepsilon]}{\left(\displaystyle\prod_{6\text{ odd }[\delta]}\theta_1[\delta]\right)^4}$$

$$J_6 = 12\pi^{36}\,\frac{\displaystyle\sum_{60\text{ terms}}\theta[\varepsilon]^8\sum_{6\text{ terms }[\varepsilon_\mathrm{k}]\neq[\varepsilon],\ \sum[\varepsilon_\mathrm{k}]=0}\ \prod_{k=1}^{6}\theta^4[\varepsilon_k]}{\left(\displaystyle\prod_{6\text{ odd }[\delta]}\theta_1[\delta]\right)^6}$$

Absolute invariants in terms of the branch points of Γ

Let $\Gamma : \{w^2 = (z-e_1)\cdots(z-e_6)\}$. Then

$$\boxed{j_1 = 144\frac{J_4}{J_2^2}, \quad j_2 = -1728\frac{(J_4J_2 - 3J_6)}{J_2^3}, \quad j_3 = 486\frac{J_{10}}{J_2^5},}$$

$$J_2 = u_0^2 \sum_{\text{fifteen}} (12)^2(34)^2(56)^2$$

$$J_4 = u_0^4 \sum_{\text{ten}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2$$

$$J_6 = u_0^6 \sum_{\text{sixty}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2(14)^2(25)^2(36)^2$$

$$J_{10} = u_0^{10} \prod_{j<k} (j,k)^2,$$

where $(i,j) = e_i - e_j$.

The puzzle of six Jacobians

Consider two genus 3 curves

$$
\begin{aligned}
C : y^2 &= P_2(x) + 2h_3\sqrt{\Phi(x)}, & \Phi(x) &= (x-c_1)(x-c_2)(x-c_3)(x-c_4), \\
K : y^2 &= P_2(x) + \sqrt{\Psi(x)}, & \Psi(x) &= P_2^2(x) - 4h_3^2\Phi(x) \\
& & &= k(x-s_1)(x-s_2)(x-s_3)(x-s_4),
\end{aligned}
$$

The coverings of even-order elliptic curves:

$$
\begin{aligned}
C &\longrightarrow E = \{z^2 = \Phi(x)\} \quad \text{ramified at} \quad Q_i = (s_i, \sqrt{\Phi(s_i)}), \\
K &\longrightarrow \mathcal{E} = \{z^2 = \Psi(x)\} \quad \text{ramified at} \quad \mathcal{P}_i = (c_i, \sqrt{\Psi(c_i)})
\end{aligned}
$$

L. Heine: Prym(C,σ) and Prym(K,σ) are *dual*.

$$
\begin{array}{ccccc}
\mathrm{Jac}(\widetilde{\Gamma}_1) & & \mathrm{Jac}(\widetilde{\Gamma}_2) & & \mathrm{Jac}(\widetilde{\Gamma}_3) \\
\| & \searrow & \downarrow & \swarrow & \| \\
\| & & \mathrm{Prym}(C,\sigma) & & \| \\
\Downarrow & \swarrow & \downarrow & \searrow & \Downarrow \\
\mathrm{Jac}(\Gamma_1) & & \mathrm{Jac}(\Gamma_2) & & \mathrm{Jac}(\Gamma_3) \\
& \searrow & \downarrow & \swarrow & \\
& & \mathrm{Prym}(K,\sigma) & & \\
& \swarrow & \downarrow & \searrow & \\
\mathrm{Jac}(\widetilde{\Gamma}_1) & & \mathrm{Jac}(\widetilde{\Gamma}_2) & & \mathrm{Jac}(\widetilde{\Gamma}_3)
\end{array}
$$

Conclusion

• We gave an *algebraic* description of the 2-dimensional Prym variety $\mathrm{Prym}(C/\sigma)$ by relating it to 6 different genus 2 curves.

• The Clebsch system, the Euler top on $SO(4)$, the Kovalevskaya top can be linearized on 3 different hyperelliptic Jacobians.

This can be applied to an explicit integration of other problems (e.g., generalizations of the Kovalevskaya top, the Neumann systems of Stiefel varieties).

• Our numerical test confirms the integration procedure of the Clebsch system made by F. Kötter.