Transcendental cases of stability problem in integrable and non-integrable Hamiltonian systems

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Problem statement

Let us consider Hamiltonian system

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}, \tag{1}$$

Suppose that

- Hamiltonian H is analytic in a small neighborhood of the origin $q = p = 0$, which is an equilibrium

$$H = H_2 + H_3 + \cdots + H_m + \ldots \tag{2}$$

- Hamiltonian H is 2π-periodically depends on t.

We investigate Lyapunov stability of the equilibrium position of system (1).

Stability in the First Approximation

Let us consider linear Hamiltonian system

$$\frac{dq}{dt} = \frac{\partial H_2}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H_2}{\partial q}, \tag{3}$$

Characteristic equation of system (3):

$$\rho^2 - 2\varkappa\rho + 1 = 0\,.$$

- If at least one of the roots $\rho_{1,2}$ of characteristic equation has modulus more then 1, then the equilibrium position of the original system with Hamiltonian H is unstable.
- If the both roots $\rho_{1,2}$ of characteristic equation has modulus less then 1, then the equilibrium position of the linear system with Hamiltonian H_2 is stable.
- Let the roots of characteristic equation be multiple and equal to 1 or -1. If the Jordan normal form of the matrizant of system (3) is diagonal, then the equilibrium position of the linear system is stable, otherwise it is unstable.

Nonlinear stability analysis

Let us assume that $|\rho_{1,2}| = 1$, i.e $\rho_{1,2} = \exp\{\pm i2\pi\lambda\}$.
Linear normalization. By means linear 2π-periodic change of variables $q, p \to x, y$ we bring the Hamiltonian into the following form

$$H = \frac{\lambda}{2}(x^2 + y^2) + H_3 + \cdots + H_m + \ldots \tag{4}$$

Nonlinear normalization. Nonresonant case $m\lambda \neq N$. By means of nonlinear close to identity 2π-periodic change of variables $x, y \to u, v$ we bring the Hamiltonian into the following normal form

$$H = \lambda r + c_2 r^2 + \cdots + c_n r^n + O(r^{n+1}), \tag{5}$$

where $u = \sqrt{2r}\sin\varphi, \quad u = \sqrt{2r}\cos\varphi$.
In nonresonant case the Arnold - Moser Theorem guaranties the stability of the equilibrium of the original system if at least one of the constants $c_2, \ldots, c_n$ is not equal to zero.

Nonlinear stability analysis

Nonlinear normalization. Resonant case. Suppose that fourth order resonance takes place, i.e. $4\lambda \neq N$
By means of nonlinear close to identity 2π-periodic change of variables $x, y \to u, v$ we bring the Hamiltonian into the following normal form

$$H = \lambda r + r^2(c_2 + a_2 \sin(4\varphi + 4\lambda t) + b_2 \cos(4\varphi + \lambda t)) + O(r^{5/2}), \qquad (6)$$

where $u = \sqrt{2r}\sin\varphi, \quad u = \sqrt{2r}\cos\varphi$.
In the case case of fourth order resonance Markeev Theorem takes place:

- If the inequality $c_2 > \sqrt{a_2^2 + b_2^2}$ is fulfilled, then the equilibrium of the original system is stable.
- If the inequality $c_2 < \sqrt{a_2^2 + b_2^2}$ is fulfilled ,then the equilibrium of the original system is stable.
- In the case $c_2 = \sqrt{a_2^2 + b_2^2}$, then an additional study of stability based on an analysis of the terms of order higher then four is necessary.

Nonlinear stability analysis

Stability or instability can be established by using the following theorems

- In nonresonant case the Arnold - Moser Theorem ;
- In resonant case (for resonances of orders higher then two) Markeev's theorems [1]
- For strong degeneration in the case of fourth order resonance Markeev formulated stability condition.
- Low order resonances the stability question is can be solved by Sokolskii's theorems.

[1] A. P. Markeev. *Libration Points in Celestial Mechanics and Space Dynamics*. Nauka, Moscow, 1978. (in Russian).

Low order resonances $\lambda = N$ or $2\lambda = N$

Suppose that a resonance of first or second order takes place, i.e. $\lambda = N$ or $2\lambda = N$.
In general position the Jordan canonical form of the matrizant of the linear system is not diagonal.
By a proper chose of canonical variables the Hamiltonian takes the form

$$H = \frac{\delta}{2}\eta^2 + \sum_{m=3}^{M} a_m \xi^m + \tilde{H}, \quad \delta = \pm 1 . \tag{7}$$

Sokolskii's Theorem [2]**.** Let us suppose that $a_k = 0$ for $k = 1, \dots, M-1$ and $a_M \neq 0$. If M is odd, then the equilibrium is unstable. If M is even, then the equilibrium is unstable for $\delta a_M < 0$ and stable for $\delta a_M > 0$.
Definition. *If for any integer m the coefficients of the normal form $a_{m,0} = 0$ for any integer m, then we shall say that a* **transcendental case** *takes place.*

[2] Sokolsky, A. G. On stability of an autonomous Hamiltonian system with two degrees of freedom under first-order resonance J. Appl. Math. Mech., 1977, 41, 20–28

Transcendental cases. Two questions.

- When transcendental cases can appear?
- Is equilibrium position of system stable or unstable in transcendental case?

Description of the problem

Consider the motion of a rigid body about the fixed point O in a uniform field of gravity.
Coordinate systems:

- OXYZ is a fixed coordinate system whose axis OZ is directed vertically upward,
- Oxyz is formed by the principal axes of inertia of the body for point O.

Goryachev-Chaplygin case:

- Principal moments of inertia (A, B and C) satisfy the following relation

$$A = C = 4B$$

- Center of mass lies in the equatorial plane of the ellipsoid of inertia at distance l from point O. We put $x_* = l, y_* = z_* = 0$.

Equation of motion

Canonical variables: Euler angles φ, θ, ψ and corresp. momenta $p_\varphi, p_\theta, p_\psi$.
ψ – a cyclic coordinate, therefore, $p_\psi = constant$. We put $p_\psi = 0$.
Introduce dimensionless variables:

$$q_1 = \varphi - \frac{3\pi}{2}, \quad q_2 = \theta - \frac{\pi}{2}, \quad p_1 = p_\varphi/(4B\mu), \quad p_2 = p_\theta/(4B\mu) \tag{8}$$

and new time $\tau = \mu t$, where $\mu^2 = mgl/(4B)$

Hamiltonian

$$\begin{aligned} H = {} & \frac{1}{2}[1 + (1 + 3\sin^2 q_1)\tan^2 q_2]p_1^2 + 3p_1p_2 \sin q_1 \cos q_1 \tan q_2 + \\ & + \frac{1}{2}(1 + 3\cos^2 q_1)p_2^2 - \cos q_1 \cos q_2 \end{aligned} \tag{9}$$

Additional first integral

$$\begin{aligned} & (p_1 \sin q_1 \tan q_2 + p_2 \cos q_1)[p_1^2 + (p_1 \cos q_1 \tan q_2 - p_2 \sin q_1)^2] + \\ & + (p_1 \cos q_1 \tan q_2 - p_2 \sin q_1) \sin q_1 \cos q_2 = \mathrm{const} \end{aligned} \tag{10}$$

Planar motions

The equations admit the partial solution:

$$q_2 = p_2 = 0, \quad \frac{dq_1}{dt} = \frac{\partial H^{(0)}}{\partial p_1}, \qquad \frac{dp_1}{dt} = -\frac{\partial H^{(0)}}{\partial q_1}, \tag{11}$$

where $H^{(0)} = p_1^2/2 - \cos q_1 = h$
On planar motion of the rigid body the principal axis of inertia Oz keeps a fixed horizontal position.

- $|h| < 1$: pendulum-like oscillations
- $|h| > 1$: pendulum-like rotations
- $|h| = 1$: asymptotic motion

Action-angle variables

In the case of **planar oscillation**

$$q_1 = 2\arcsin[k_1 \,\mathrm{sn}(u, k_1)], \quad p_1 = 2k_1 \,\mathrm{cn}(u, k_1), \quad u = 2\pi^{-1} K(k_1) w \qquad (12)$$

where $k_1 = k_1(I)$ is the inverse of the function

$$I = 8\pi^{-1}[E(k_1) - (1 - k_1^2)K(k_1)]. \qquad (13)$$

In the case of **planar rotation**

$$q_1 = 2\mathrm{am}(u, k_2), \quad p_1 = 2k_2^{-1} \,\mathrm{dn}(u, k_2), \quad u = \pi^{-1} K(k_2) w, \qquad (14)$$

where $k_2 = k_2(I)$ is the inverse of the function

$$I = 4E(k_2)/(\pi k_2). \qquad (15)$$

Perform the canonical change of variables $q_1, p_1, q_2, p_2 \to I, w, q_2, p_2$.

Perturbed motion

In the phase space of the canonical variables I, w, q_2, p_2 the unperturbed periodic motions are described by the following family of periodic orbit.

$$I = I_0 = \mathrm{const}, \qquad w = \omega\tau + w(0), \qquad q_2 = p_2 = 0. \tag{16}$$

where ω is the frequency of unperturbed periodic motion.
Introduce the perturbation $r_1 = I - I_0$ of the variable I and expand the Hamiltonian $\Gamma(w, r_1, q_2, p_2)$ into a power series in r_1, q_2, p_2

$$\Gamma = \Gamma_2 + \Gamma_4 + \dots + \Gamma_{2m} + \dots\ , \tag{17}$$

where Γ_{2m} is a form of degree $2m$ in variables $q_2, p_2, |r_1|^{1/2}$ with coefficients which are periodic functions of w.

$$\Gamma_2 = \omega r_1 + \Gamma_2^{(0)}(q_2, p_2, w), \quad \Gamma_2^{(0)} = f_{20}q_2^2 + f_{11}q_2p_2 + f_{02}p_2^2, \tag{18}$$

$$\begin{aligned} f_{20} &= \frac{1}{4}\left[(2\alpha - (2\alpha - 1)\cos^2 q_1)p_1^2 + 2\cos q_1\right], \\ f_{11} &= \frac{1}{2}(2\alpha - 1)p_1 \sin q_1 \cos q_1, \ \ f_{02} = \frac{1}{4}\left[(2\alpha - 1)\cos^2 q_1 + 1\right], \end{aligned} \tag{19}$$

Isoenergetic reductuon

Let us consider the motion on the energy level

$$\Gamma = 0 \tag{20}$$

By solving (20) we have $r_1 = -K(q_2, p_2, w)$. The motion on the isoenergetic level $\Gamma = 0$ is described by Whittaker equations

$$\frac{dq_2}{dw} = \frac{\partial K}{\partial p_2}, \quad \frac{dp_2}{dw} = -\frac{\partial K}{\partial q_2}. \tag{21}$$

The problem of the orbital stability is reduced to the stability problem for the equilibrium position $q_2 = p_2 = 0$ of the reduced system (21). The reduced system has the following first integral

$$G_1 + O_3 = \mathrm{const} \tag{22}$$

the linear part

$$G_1 = p_1 \sin q_1 (p_1^2 + \cos q_1) q_2 + (p_1^2 \cos q_1 - \sin^2 q_1) p_2 \tag{23}$$

where q_1 and p_1 in (23) are functions of w, corresponding to the unperturbed motion.

Linear system

$$\frac{dq_2}{dt} = \frac{1}{\omega}(f_{11}q_2 + 2f_{02}p_2), \qquad \frac{dp_2}{dt} = -\frac{1}{\omega}(2f_{20}q_2 + 2f_{11}p_2) \tag{24}$$

Matrizant reads

$$X(T) = \left\| \begin{matrix} 1 & x_{12}(T) \\ 0 & 1 \end{matrix} \right\| \tag{25}$$

In the case of oscillations

$$x_{12}(T) = x_{12}(\pi) = 2\frac{(4k_1^2 - 2)E(k_1) - (3k_1^2 - 2){k_1'}^2 K(k_1)}{{k_1'}^4} \tag{26}$$

In the case of rotations

$$x_{12}(T) = x_{12}(2\pi) = 2\frac{k_2({k_2'}^2 K(k_2) - 2(2 - k_2^2)E(k_2))}{{k_2'}^4} \tag{27}$$

where ${k_i'}^2 = 1 - k_i^2$ $(i = 1, 2)$.
Characteristic equation are multiple root $\rho = -1$, i.e. second order resonance takes place.

Normalization

By means linear change of variables $q_2, p_2 \to \xi, \eta$ we bring the Hamiltonian into the following form

$$F = \frac{1}{2}\eta^2 + O_4 \tag{28}$$

In variables ξ, η the additional first integral reads

$$\eta + \tilde{G}^{(3)}(\xi, \eta, w) = \mathrm{const} \tag{29}$$

By means linear change of variables $\xi, \eta \to x, y$ we bring the Hamiltonian into form

$$N = \frac{1}{2}y^2 + a_{2n}x^{2n} + O_{2k+1} \tag{30}$$

Calculations shown that $a_4 = a_6 = a_8 = 0$. We suspect that $a_{2n} = 0$ for any integer n, i.e. transcendental case takes place.

Proof of instability

Now we construct canonical (and close to identity) change of variables which brings the first integral into the form

$$P = \mathrm{const} \tag{31}$$

Let $S(\xi, P, w)$ be a generating function of such a change of variables

$$Q = \frac{\partial S}{\partial P}, \quad \eta = \frac{\partial S}{\partial \xi} \tag{32}$$

On the other hand, we have the relation

$$\eta + \tilde{G}^{(3)}(\xi, \eta, w) = P \tag{33}$$

For small ξ and P we can solve (33) with respect to η

$$\eta = P + \Psi(\xi, P, w) \tag{34}$$

where $\Psi(\xi, P, w)$ – T-periodic in w, analytic with respect to ξ and P function
Thus, we have the equation for S

$$\frac{\partial S}{\partial \xi} = P + \Psi(\xi, P, w) \tag{35}$$

By solving (35), we have

$$S = \xi P + \int_0^\xi \Psi(y, P, w) dy + f(P, w) \tag{36}$$

where $f(P, w)$ is an arbitrary function. We put $f(P, w) \equiv 0$.

Does the transcendental case take place?
Yes!

We give the proof by contradiction.
Let us suppose that $a_{2n} \neq 0$ for certain integer n.

- From Sokolskii theorem it follows that $a_{2n} < 0$, otherwise we have stability.
- In this case there exist two families of asymptotic solutions G.A. Merman has shown [3] tending to equilibrium as $w \to \infty$.
- Asymptotic solutions can exist only on the manifold of zero-level of first integral ($P = 0$).
- The above manifold includes only the family of equilibriums: $Q = costant, \quad P = 0$.

[3] G.A. Merman Asymptotic solutions of canonical system with one degree of freedom in the case of zero characteristic exponents Bul. Inst. of Theoretical Astronomy, 1964. V 9., No. 6, pp. 394 – 424.

When transcendental cases can appear in periodic Hamiltonian system?

$$H = \frac{\delta}{2}\eta^2 + a_M\xi^M + \tilde{H}(\eta, \xi, t), \quad \delta = \pm 1\,. \tag{37}$$

G.A. Merman has shown [4] that in transcendent case 2π-periodic Hamiltonian system with one degree of freedom posses one parametric family of 2π-periodic solutions emanating from the equilibrium. Thus, the existence of family of 2π-periodic solutions is a necessary condition for the transcendent case.
This condition is also sufficient. Let us suppose that 2π-periodic Hamiltonian system admit a family of solution of the following form

$$q = g(t, \alpha), \quad p = f(t, \alpha), \tag{38}$$

where $g(t, \alpha)$, $f(t, \alpha)$ are 2π-periodic functions of variable t, and analytically depend on parameter α. Moreover, $g(t, 0) = f(t, 0) \equiv 0$
Let us perform canonical 2π-periodic change of variables $q, p \to \alpha, p_\alpha$ such that, in α, p_α solution (38) reads

$$\alpha = const, \quad p_\alpha = 0, \tag{39}$$

The generating function $S(p, \alpha)$ can be found from the condition

$$q = -\frac{\partial S}{\partial p}, \quad p_\alpha = -\frac{\partial S}{\partial \alpha}\,. \tag{40}$$

[4] G.A. Merman Asymptotic solutions of canonical system with one degree of freedom in the case of zero characteristic exponents Bul. Inst. of Theoretical Astronomy, 1964. V 9., No. 6, pp. 394 – 424.

Stability study in the transcendental case

The calculations have shown

$$S = -pg(\alpha, t) + \int_0^\alpha \frac{\partial g}{\partial u}(u, t)f(u, t)du \tag{41}$$

In new variables α, p_α the Hamiltonian takes form

$$\Gamma = p_\alpha^2 F(\alpha, p_\alpha, t) \tag{42}$$

where

$$F = \frac{1}{2} + \sum_{n=k}^{\infty} \sum_{i+j=n} f_{ij}\alpha^i p_\alpha^j \tag{43}$$

The structure of the Hamiltonian (42) guarantee, that the coefficient a_M of the normal form for (37) are equal to zero for any M. Thus the following theorem is proven.

Lemma. *If the canonical system with Hamiltonian* (37) *has one parametric family of* 2π*-periodic solutions, emanating from the equilibrium, then the transcendental case takes place and equilibrium position is unstable.*

Thank you for your attention!